Vol. 42, No. 3 DUKE MATHEMATICAL JOURNAL© September 1975

HYPERCONTRACTIVITY AND LOGARITHMIC SOBOLEV INEQUALITIES FOR THE CLIFFORD-DIRICHLET FORM

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1. Introduction. Recent developments in the constructive theory of quantum fields have led to new types of Sobolev inequalities. For example, the logarithmic Sobolev inequality [9]

$$1.1)\qquad \int_{R^n} |f|^2 \ln |f| \, d\nu \leq \int_{R^n} |\text{grad}\, f|^2 \, d\nu + ||f||_2^2 \ln ||f||_2 \, ,$$

where ν denotes Gauss measure on R^n and $||f||_2$ denotes the $L^2(\nu)$ norm, is an outgrowth of ideas beginning with E. Nelson's proof [12] of semi-boundedness of the total Hamiltonian in a particular model for quantum field theory. In [9] we showed that if N is the self adjoint operator corresponding to the Dirichlet form for ν, that is,

$$1.2)\qquad (Nf, g)_{L^2(\nu)} = \int_{R^n} \text{grad}\, f \cdot \text{grad}\, \bar{g} \, d\nu,$$

then the inequality 1.1) is equivalent to the family of inequalities (known as hypercontractivity inequalities)

$$1.3)\qquad ||e^{-tN}f||_p \leq ||f||_q \text{ if } e^{-2t} \leq (q-1)/(p-1) \text{ and } f \text{ is in } L^q(\nu) \text{ and } t \geq 0.$$

The inequalities 1.3), which are due to Nelson [13, 14] represent a culmination of improvements due to Glimm [7] and Segal [18] of Nelson's original inequalities [12], which are of a similar nature.

Logarithmic Sobolev inequalities similar to 1.1), but with ν replaced by a non Gaussian measure, have been proved by J. P. Eckmann [4] on R^n and by J. Rosen [16] on R^1. The non Gaussian measure is that associated with the ground state of a Schrödinger operator. Both authors explore the connection between the logarithmic Sobolev inequality and hypercontractivity properties of the associated semi-group. Logarithmic Sobolev inequalities involving higher order derivatives have been investigated by G. Feissner [6]. Very illuminating connections between various kinds of Sobolev inequalities on the one hand and the Heisenberg uncertainty relations on the other have been explored by W. Faris [5].

Finally, we mention that W. Beckner [1] has adapted the techniques from [9], which we used to prove 1.1), to give a very novel proof of a strengthened version of the Hausdorff-Young inequality.

Received January 27, 1975. J. S. Guggenheim Memorial Foundation Fellow. This research was partially supported by N.S.F. grant GP 31380.