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ISOMETRIES OF H^∞

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If D is a bounded open set in the complex plane which is a "maximal" domain for the space $H^\infty(D)$ of bounded holomorphic functions in D, then it is shown in [15] that any surjective isometry, Φ, of $H^\infty(D)$ is of the form $\Phi(f) = \alpha \cdot f \circ \varphi$ where $|\alpha| = 1$ and φ is a conformal mapping of D onto itself. In this note we consider the surjective isometries of $H^\infty(D)$ where D is a bounded domain in $\mathbf{C}^N$. It follows from general considerations, see [7], that any surjective isometry Φ of $H^\infty(D)$ is of the form $\Phi(f) = \alpha \cdot \Phi_1(f)$ where $|\alpha| = 1$ and Φ_1 is an algebra automorphism of $H^\infty(D)$, so in what follows we will consider only automorphisms Φ of $H^\infty(D)$. It also follows from general considerations that if Φ is an automorphism of $H^\infty(D)$ then $\Phi f = \hat{f} \circ \varphi$ where φ is a homeomorphism of the maximal ideal space of $H^\infty(D)$ and $\hat{f}$ is the Gelfand transform of f. Hence we are looking for conditions on D that will imply that $\varphi(\mathrm{D}) = D$ and that φ is biholomorphic on D.

If D_1 is a domain containing D and every function $f \in H^\infty(D)$ extends to be holomorphic in D_1 then any biholomorphic mapping of D_1 onto itself will give rise to an automorphism of $H^\infty(D)$, so the problem of finding the domains in $\mathbf{C}^N$ that are maximal for bounded holomorphic functions is clearly related to the automorphism question. We discuss this in the first section for $N = 1$. For $N > 1$ the problem is still open. We give some examples, discuss differences between the cases $N = 1$ and $N > 1$, and give an example which answers in the negative a question of Kobayashi [10] concerning the Carathéodary metric. In the second section we describe a class of domains D in $\mathbf{C}^N$ for which every automorphism Φ of $H^\infty(D)$ is of the form $\Phi f = f \circ \varphi$ where φ is a biholomorphic mapping of D onto itself. This class includes the strictly pseudoconvex domains with smooth boundary and the non-degenerate Oka-Weil domains.

1. Let D be a bounded connected domain in $\mathbf{C}^N$. If $F \subseteq H^\infty(D)$ and $K \subseteq D$ we define

$$\hat{K}_F = \{z \in D : |f(z)| \leq ||f||_K \text{ for all } f \in F\}.$$

where $||f||_K$ denotes the supremum of f on K. We say that D is F-convex if whenever K is compact, $\hat{K}_F$ is compact. We say that D is a domain of bounded holomorphy if for every $a \in \partial D$ there is a function $f \in H^\infty(D)$ that has no analytic continuation through a (see [16] for a more precise statement). In

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