CONTINUITY PROPERTIES OF POTENTIALS

NORMAN G. MEYERS

Introduction. Let $k = k(x)$ be a continuous, non-negative, and radially symmetric kernel on Euclidean n-space, which is non-increasing in $|x|$. The present work is concerned with the continuity properties of $u = k * f$ where f is a non-negative function in the Lebesgue space $\mathfrak{L}_p$, $1 < p < \infty$. Bessel and Riesz potentials are important special cases.

The function u is lower semi-continuous but may be continuous nowhere. However, if we restrict the set of approach to x_0, we may be able to conclude

$$u(x_0) = \lim_{\substack{x \to x_0 \\ x \notin E}} u(x).$$

The question is: If we are willing to allow the excluded set E to have only a certain size at x_0 for which points x_0 can such an assertion be made? Theorems 3.1 and 3.2 are two examples of such results which are based on a simple and general method outlined at the end of section 3. Though part of the conclusion of Theorem 3.1 can be derived by the traditional method based on the fine topology associated with the capacity potentials (see [9]), the advantage of our method is that it gives an explicit and simple condition for 'continuity' at a point. Theorem 2.1 is a prerequisite for Theorem 3.1 in that it tells us on how large a set this condition, which concerns the variation of Radon measures over balls, must hold.

1. Preliminaries. All our work will be carried out in the Euclidean space of ordered n-tuples of real numbers, R^n.

$\mathfrak{M}$ will denote the space of all real Radon measures on R^n. We adopt the practice of denoting positive elements in a space by a superscript plus. Thus $\mathfrak{M}^+$ is the cone of all positive Radon measures. We recall that every element of $\mathfrak{M}^+$ can be identified with a measure μ, which is the completion of a positive Borel measure which is finite on compact sets. We let $\mathfrak{A}$ be the σ-algebra of all sets which are measurable for *every* $\mu \in \mathfrak{M}^+$. If $A \in \mathfrak{A}$ and $\mu \in \mathfrak{M}^+$ we say that A is a *carrier* of μ if $\mu(A_0) = 0$ for all $A_0 \in \mathfrak{A}$ with $A_0 \cap A = \emptyset$. Let $\mu \in \mathfrak{M}$ and let f be a μ-measurable function defined μ-a.e. with values in $[-\infty, +\infty]$. If $\int_K f(x)\, d\mu(x)$ exists and is finite for every compact K we call the corresponding Radon measure $f \cdot \mu$. If $\mu = m$ = Lebesgue measure we follow the common practice of identifying f with $f \cdot m$. This is consistent with our general practice of dropping m from notations; thus $dx = dm(x)$.

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