SIMPLICITY OF THE C*-ALGEBRA ASSOCIATED WITH THE FREE GROUP ON TWO GENERATORS

ROBERT T. POWERS

Introduction. At a conference held in Baton Rouge in 1967 J. Dixmier posed the question of whether every simple C^*-algebra is generated by its projections. A couple of years later R. Kadison suggested to us that the C^*-algebra associated with the left regular representation of the free group on two generators should provide an example of a simple C^*-algebra without projections. In this paper we show that this algebra is simple, i.e., it has no non trivial two-sided ideals. We still do not know if this algebra contains projections.

Notations and definitions. Let $\mathfrak{F}$ be the free group on two generators a and b. The elements $g \in \mathfrak{F}$ often called words, are expressions of the form $a^{n_1}b^{m_1}a^{n_2} \cdots b^{n_r}$ or $b^{m_1}a^{n_1}b^{m_2} \cdots b^{m_r}$ where n_i, $m_j = 0, \pm 1, \pm 2, \cdots$. A word is in reduced form if all the n_i and m_j are not zero. To multiply two group elements g_1 and g_2 one writes the combined word $g_1 g_2$ and then reduces this word if necessary, e.g., $(aba^2)(a^2b^{-1}a) = aba^4b^{-1}a$ and $(aba^2)(a^{-2}b^{-1}a) = a^2$.

Let $\mathfrak{H} = L^2(\mathfrak{F})$ be the Hilbert space of all complex valued functions f on $\mathfrak{F}$ such that $\sum_{g \in \mathfrak{F}} |f(g)|^2 < \infty$. We use the physicist's inner product on $\mathfrak{H}$ (which is linear in the second factor) given by

$$(f, h) = \sum_{g \in \mathfrak{F}} \overline{f(g)} h(g).$$

For each $g_1 \in \mathfrak{F}$ we define the unitary operator $U(g_1)$ on $\mathfrak{H}$ given by

$$(U(g_1)f)(g) = f(g_1^{-1}g) \quad \text{for all} \quad f \in \mathfrak{H}.$$

One can easily show that $g \to U(g)$ is a unitary representation of $\mathfrak{F}$ on $\mathfrak{H}$. This representation is called the left regular representation of $\mathfrak{F}$ on $L^2(\mathfrak{F})$.

Let $\mathfrak{A}_0(\mathfrak{F})$ be the $*$-algebra of operators A on $\mathfrak{H}$ of the form

$$A = \sum_{i=1}^{n} \alpha_i U(g_i)$$

with α_i complex numbers and $g_i \in \mathfrak{F}$ and $n = 1, 2, \cdots$. Let $\mathfrak{A}(\mathfrak{F})$ be the C^*-algebra formed by taking the closure of $\mathfrak{A}_0(\mathfrak{F})$ in the operator norm topology. We will prove that $\mathfrak{A}(\mathfrak{F})$ is simple.

Let $e_0 \in L^2(\mathfrak{F})$ be the function which is one at the identity e and zero off the identity, i.e., $e_0(e) = 1$ and $e_0(g) = 0$ for $g \neq e$. Let τ be the state on $\mathfrak{A}(\mathfrak{F})$

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