RINGS WITH AN INTEGRAL ELEMENT WHOSE CENTRALIZER SATISFIES A POLYNOMIAL IDENTITY

MARTHA SMITH

1. Introduction. S. Montgomery [5] has recently shown that if the centralizer of a suitable algebraic element of a ring satisfies a polynomial identity (PI), then the ring must be PI. In this paper we extend these results to a more general setting.

Montgomery imposed the conditions that an element a have some power a^n lying in the center, where both a and n are invertible in the ring. The invertibility of a enabled her to work with the inner automorphism induced by a. We drop this assumption and work instead with the derivation induced by a. Together, the invertibility of a and n form a sort of separability condition. We replace this in Section 2 by a more general separability condition described in the next paragraph.

Let Ω be a commutative integral domain with identity, R an Ω-algebra and $a \in R$ integral over Ω. Let D denote the derivation $Dx = ax - xa$ on R (or any other ring containing a). Since D is the difference of the left and right multiplications by a, which are both integral over Ω and commute, D must also be integral over Ω. We will assume in most of Section 2 that D satisfies a polynomial $f(\lambda) \in \Omega[\lambda]$ such that $f'(0)$ acts invertibly on R. This may naturally be regarded as a separability condition, as the following Lemma shows.

LEMMA 1. *Let Ω, R, a, D be as above. Suppose Ω is a field and f is the minimum polynomial of D over Ω. If a is separable over Ω, then $f'(0) \neq 0$. If R is semiprime and the minimum polynomial of a splits in Ω, then $f'(0) \neq 0$ implies a is separable over Ω.*

Proof. Let $\bar{\Omega}$ be any field extension of Ω, $R' = \bar{\Omega} \otimes_{\Omega} R$. It is easy to see that $1 \otimes a$ and $1 \otimes D$ have the same minimal polynomials over $\bar{\Omega}$ as a and D, respectively, do over Ω. Thus for the first part we may without loss of generality suppose the minimal polynomial of a splits in Ω. Let $\alpha_1, \cdots, \alpha_n$ be the roots of this minimal polynomial. Then since a is separable

$$R = \sum_{i,j=1}^{n} \oplus R_{ij} ,$$

a direct sum as Ω-vector spaces, where for $x \in R_{ij}$, $ax = \alpha_i x$, $xa = \alpha_j x$. Suppose $D^2 r = 0$. Write $r = \sum r_{ij}$, $r_{ij} \in R_{ij}$. Then

$$0 = D^2 r = \sum (\alpha_i - \alpha_j)^2 r_{ij}$$

implies $r_{ij} = 0$ except when $i = j$. Thus $Dr = 0$.

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