SOME RESULTS ON REGULARITY FOR SOLUTIONS OF NON-LINEAR ELLIPTIC SYSTEMS AND QUASI-REGULAR FUNCTIONS

NORMAN G. MEYERS AND ALAN ELCRAT

Introduction. We are concerned here with non-linear elliptic systems in variational form, of order $2m$ and in N unknown functions; solutions are assumed to be in the Sobolev space $W^{m,p}(\Omega)$, p $(1 < p < \infty)$ being the natural exponent for the system.

Theorem 1 shows that, in fact, a solution is in $W_{loc}{}^{m,p+\epsilon}(\Omega)$, where ϵ is a positive number depending only on the structure of the system. This result is of particular interest if $(m - j)p = n$, the dimension of the underlying Euclidean space. For then, by the Sobolev imbedding theorem, the solution must have Hölder continuous derivatives up to the order j. The interest in the above consequence derives from the fact that when $m > 1$ or $N > 1$ the solution need not be continuous. We have in mind the examples of Di Giorgi [2], Giusti and Miranda [5] and Frehse [3]. Frehse, in particular, shows that if $mp < n$, there is an equation with a discontinuous solution, in the case $m = 1$, $N = n$ and in the case $m = 2$, $N = 1$. Widman [12] also has proved the Hölder continuity of solutions in the case $mp = n$. Widman's proof is based on the rate of growth of the p-norm of the m-th order derivatives over balls. Such a rate of growth follows immediately from Theorem 1.

As a further consequence of Theorem 1 we show that a quasi-regular mapping is in $W_{loc}{}^{1,n+\epsilon}$, where ϵ depends only on n and on the distortion constant. This had been shown for one to one maps (quasi-conformal) by Gehring [4].

The main tool in the proof of Theorem 1 is a result of Gehring, which states that when a certain reverse maximal function inequality holds for a function in L^1, the function is, in fact, in $L^{1+\epsilon}$.

In Theorem 2 we show that under mild regularity conditions on $\partial\Omega$, given in terms of Bessel capacities, a solution in $W_0{}^{m,p}(\Omega)$ is in $W^{m,p+\epsilon}(\Omega)$; if $p > n$ no condition on Ω is needed.

Finally, in the last section we consider linear systems. Here the natural class in which to consider a solution is $W^{m,2}(\Omega)$. We show by a duality argument that if a solution is in $W_0{}^{m,p/p-1}(\Omega)$, where $2 < p$ is not too large, then the solution is in $W_0{}^{m,p}(\Omega)$. The case $m = 1$, $N = 1$ was proved in [6]. Examples in [11] show that if p is too large this is false and the solution may exhibit pathological behavior.

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