APPROXIMATION BY ANALYTIC FUNCTIONS UNIFORMLY CONTINUOUS ON A SET

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Let D be a bounded plane domain and let $F \subset D$ be a relatively closed subset. We shall consider the following classes of functions.

$$H(D) = \{f: f \text{ holomorphic in } D\},$$

$$U(F) = \{f \in H(D): f \text{ is uniformly continuous on } F\},$$

$$B(F) = \{f \in H(D): f \text{ is bounded on } F\}.$$

If J is a compact subset of the plane, then

$$A(J) = \{f \in C(J): f \text{ is holomorphic on } J^\circ\},$$

where J° denotes the interior of J.

Let $K = F^-$ (the closure of F). The functions in $U(F)$ may be regarded as being defined on K. Thus

$$U(F)|_K \subset C(K).$$

The left side is an algebra but in general it is not closed. The goal of this paper is to describe its closure. To do this we need the notion of the hull of F with respect to the family $U(F)$. The supremum of f on a set S will be denoted by $\|f\|_S$, even when f is unbounded on S.

$$\text{Hull } (F) = \{z \in K \cup D: |f(z)| \leq \|f\|_K,\ f \in U(F)\}.$$

Our main result is that, with some restriction on D, the closure of $U(F)|_K$ in $C(K)$ may be identified with $A(\text{Hull } (F))$. This will hold, in particular, when D is the unit disc.

LEMMA 1. *Hull (F) is compact.*

Proof. Hull (F) is bounded, so we must show that it is closed. Clearly any limit points that lie in D are in the hull. Now let $z_0 \in \partial D$ be a limit point of Hull (F). Since $K \subset$ Hull (F), we must show $z_0 \in K$. Assume that this is false. Then $f(z) = 1/(z - z_0)$ is in $U(F)$. There is a neighborhood G of z_0 such that $|f| > \|f\|_K$ throughout $G \cap D$, which is a contradiction.

One can also define the hull of F with respect to the family $B(F)$:

$$\text{Hull}_B\ (F) = \{z \in D: |f(z)| \leq \|f\|_F,\ \forall\ f \in B(F)\}.$$

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