PURELY IMAGINARY SCATTERING FREQUENCIES FOR EXTERIOR DOMAINS

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1. Introduction. In this paper we study the distribution of purely imaginary poles of the scattering matrix for waves scattered by an obstacle. It was shown by Lax and Phillips in [6] that the size of an obstacle $\mathcal{O}$ in euclidean space $\mathbf{R}^3$ may be estimated in terms of the asymptotic behavior of such poles as follows: Let $N(\sigma)$ denote the number of poles on the line segment $[0, i\sigma]$, $\sigma > 0$, of the scattering matrix for the wave equation in the exterior of $\mathcal{O}$ with Dirichlet or Neumann boundary condition. Then if $\mathcal{O}$ contains a ball of radius R_1, $\liminf N(\sigma)/\sigma^2 \geq (R_1/\gamma_0)^2/2$ as $\sigma \to \infty$, where the constant $\gamma_0 = .66274 \cdots$. Furthermore, if $\mathcal{O}$ is star-shaped and contained in a ball of radius R_2, then $\limsup N(\sigma)/\sigma^2 \leq (R_2/\gamma_0)^2/2$. We show here that this result can be extended to obstacles in $\mathbf{R}^n$, $n \geq 3$ and odd, with general Robin boundary condition. The corresponding inequalities are $\liminf N(\sigma)/\sigma^{n-1} \geq (R_1/\gamma_0)^{n-1}/(n-1)!$ and $\limsup N(\sigma)/\sigma^{n-1} \leq (R_2/\gamma_0)^{n-1}/(n-1)!$ If $\mathcal{O}$ is star-shaped, the limits are independent of the boundary condition. With a certain restriction, the imaginary poles sufficiently far out depend monotonically on the obstacle, or if the obstacle is fixed, on the boundary condition. A similar result is obtained for Maxwell's equations with the boundary condition of a perfect conductor. For the wave equation in even dimensional space there are at most a finite number of imaginary poles.

The scattering matrix for an obstacle in odd dimensions is a certain operator-valued meromorphic function on the complex plane which measures the deflection by the obstacle of waves approaching it from infinity. In many physical interactions it is the natural observable quantity. The poles of this function lie in the upper half-plane and correspond to poles of the "outgoing" Green's function for the reduced wave equation on the exterior domain. They are in a sense generalized eigenvalues: if z is a pole, there is a solution of the wave equation of the form $e^{izt}u(x)$ satisfying an appropriate homogeneous boundary condition on $\partial\mathcal{O}$ and a growth condition at infinity. If z is imaginary, this solution decays exponentially in time without oscillation. The function u is called a scattering eigenfunction.

The general outline of the approach used here is as follows. For the wave equation, the scattering matrix $\mathcal{S}(z)$ is a linear operator on $L_2(S^{n-1})$, the Hilbert space of square integrable functions on the unit sphere in $\mathbf{R}^n$, of the form identity plus an integral operator $K(iz)$. $K(iz)$ is closely related to another integral operator $k(iz)$, called the transmission coefficient, defined in terms of the angular behavior at infinity of scattered plane waves. $\mathcal{S}$ has a pole at z if and only if

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