SOME SINGULAR SCHRÖDINGER OPERATORS WITH DEFICIENCY INDICES (n^2, n^2)

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1. Introduction. Recently several authors have formulated criteria for the essential self-adjointness of several overlapping classes of differential operators with singular coefficients. Specifically Kalf–Walter [10], Schmincke [11], Kato[12] and Simon [13], [14] have formulated such criteria and they call their potentials strongly singular. They illustrate the failure of essential self-adjointness if their requirement on the potential is violated by means of an example of a spherically symmetric potential. For the Dirac operator a similar example was given, independently, by Brownell [9].

In this paper the question of the failure of essential self-adjointness is taken up again. We show that for each pair of integers of the form (n^2, n^2) there is a real number c such that the formal operator

$$L(c) = -\Delta + \frac{c}{r^2} \quad in \quad \mathfrak{L}_2(\mathfrak{R}_3)$$

defines a symmetric operator with deficiency indices (n^2, n^2). This is described in specific terms in Theorem 2.1. The proof of this theorem makes essential use of the well known fact [16] that this operator admits a complete family of reducing subspaces on each of which it acts like an ordinary differential operator. In Section 3 we consider an abstract version of this situation. Namely an abstract Hilbert space $\mathfrak{A}$ and a symmetric operator A acting in the Hilbert space tensor product $\overline{\mathfrak{L}_2(\mathfrak{R}^+) \otimes \mathfrak{A}}$. We assume that the operator A admits a complete family of reducing subspaces of the form $\ddot{\mathfrak{C}}_\infty(\mathfrak{R}^+) \otimes \mathfrak{C}(l)$ and that on each of them it equals the Kronecker product $A(l) \otimes I$. Here $\ddot{\mathfrak{C}}_\infty(\mathfrak{R}^+)$ denotes the class of infinitely differentiable functions on $\mathfrak{R}^+$ which vanish near zero and infinity. Then in Lemma 3.1 we give a formula for the deficiency indices of A in terms of those of $\{A(l)\}$. This formula is as to be expected and the proof is straightforward. Since we have been unable to find a shorter one we included it. One of the difficulties is to show that for these operators the adjoint of the Kronecker product equals the closure of the Kronecker product of the adjoints. In Section 4 we apply this formula to the operator $L(c)$ of Theorem 2.1. In evaluating the deficiency indices of the corresponding ordinary differential operators we employ series expansions around each of the two singular points of these operators. The use of such expansions in connection with the study of deficiency indices was recently emphasized by Simon [13].

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