ON ISOMETRIES BETWEEN TEICHMÜLLER SPACES

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Introduction. In an important paper [7], H. L. Royden showed that every biholomorphic self-mapping of the Teichmüller space $T(g, 0)$, $g \geq 2$, is an element of the modular group Mod $(g, 0)$. In [3] these authors solved the corresponding problem for the Teichmüller spaces $T(g, n)$. (The answer was already known to Royden—oral communication.)

The method of [7] and [3] is to prove first that every such biholomorphic map is an isometry in the Teichmüller metric and then to study holomorphic isometries between Teichmüller spaces. Our purpose here is to study all isometries between Teichmüller spaces, with no differentiability assumptions at all. Our principal result is the following generalization of the results of [7] and [3].

MAIN THEOREM. *Let $T(g, n)$ and $T(g', n')$ be Teichmüller spaces with $2g + n > 4$ and $2g' + n' > 4$. If U is a domain in $T(g, n)$ and f: $U \to T(g', n')$ is an isometry with open range, then $(g', n') = (g, n)$ and f is the restriction to U of an element of the extended modular group* Mod $(g, n)^{\sim}$. *Thus f is either holomorphic or conjugate holomorphic.*

We shall prove this theorem in §4. Sections 1 and 3 study the linear isometries between spaces of quadratic differentials. In Section 2 we study holomorphic isometries between open sets in possibly distinct Teichmüller spaces, generalizing a theorem of D. B. Patterson [6].

1. Complex Linear Isometries.

1.1. Throughout this paper S will denote a Riemann surface of finite type (g, n) with

$$2g - 2 + n > 0 \tag{1.1.1}$$

and $\bar{S}$ its compactification. (Recall that g is the genus of the closed Riemann surface $\bar{S}$, and n is the number of points in $\bar{S}\backslash S$. These n points are called the punctures of S.) Denote by $Q(S)$ the Banach space of L^1 holomorphic quadratic differentials on S. The norm on $Q(S)$ is given by

$$||\varphi|| = \frac{1}{2}\int_S |\varphi|, \qquad \varphi \in Q(S). \tag{1.1.2}$$

The differentials in $Q(S)$ are meromorphic on $\bar{S}$, with at worst a simple pole at

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