PID'S WITH SPECIFIED RESIDUE FIELDS

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When can a collection $\mathfrak{F}$ of fields be the collection of all residue fields of some principal ideal domain D (counting multiplicity)? We restrict our attention to the case that D has characteristic 0, since then D can have residue fields of all characteristics.

One obvious restriction is (CP): For each characteristic $p \neq 0$, $\mathfrak{F}$ contains only finitely many fields of that characteristic (because the factorization of $p = p \circ 1_D$ in D involves only finitely many primes of D, which are all of those producing residue fields of characteristic p). There is no corresponding restriction for characteristic 0 (e.g., $D = \mathbf{Q}[x]$). For countable D, we show that (CP) is the only restriction.

THEOREM A. *Let $\mathfrak{F}$ be any countable collection of countable fields satisfying* (CP). *Then there is a countable* PID *of characteristic zero whose collection of residue fields is precisely $\mathfrak{F}$.*

One can be more ambitious by trying to find a Dedekind domain with prescribed class group and residue fields. A partial result in this direction forms the second main result of this paper.

THEOREM B. *Let G be a countable abelian torsion group. Then there is a countable Dedekind domain of characteristic* 0 *whose class group is G, and whose residue fields are those of the integers (i.e., one copy of $\mathbf{Z}/p\mathbf{Z}$ for each prime p).*

We now outline the proof of Theorem A. It utilizes the fact that if F_0 is the prime field of a countable field F, then there is a sequence of subfields $F_0 \subset F_1 \subset F_2 \subset \cdots$ whose union is F and which has the property that F_n is either a simple algebraic or a simple transcendental extension of F_{n-1}. The construction proceeds in three stages.

The Initial Step. Construct a PID whose residue fields are one copy of each $\mathbf{Z}/p\mathbf{Z}$ and infinitely many copies of $\mathbf{Q}$. Thus we obtain all the necessary prime fields—except for (finite) multiplicity. (Any unwanted prime field may be removed via localization.)

The Induction Step. Given a PID D with quotient field K, construct a PID $\tilde{D}$ between $D[x]$ and $K(x)$ such that the residue fields of $\tilde{D}$ are the same as those of D except that one of the residue fields of D has been altered by being replaced by

(a) two copies of itself; or

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