FACTORS OF SOME DIRECT PRODUCTS

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For any positive integer m let N_m be the set of m integers, $0, 1, 2, \cdots, m-1$. Let N be the set of nonnegative integers. A set $C \subseteq N \times N$ is a *cylinder* if (a, b) is in C if and only if $(a, 0)$ and $(0, b)$ are in C. A cylinder may be thought of as the direct product of its intersections with the two axes. If A, B, and S are subsets of $N \times N$ and each element of S is uniquely expressible in the form $a + b$, where $a \in A$ and $b \in B$, then (A, B) is called a *factoring* of S, and we write $S = (A, B)$.

In [2] Hansen proved that if $N_m \times N_n = (A, B)$, then A and B are cylinders. We shall obtain the following generalization, using a different argument.

THEOREM 1. *If C is a finite cylinder and $C = (A, B)$, then A and B are cylinders.*

Furthermore, Theorem 1 and its proof generalize directly to higher-dimensional space.

In [3] Niven proved that if $(A, B) = N \times N$, then A is a cylinder if and only if B is a cylinder. We generalize this as follows.

THEOREM 2. *If C is a cylinder and $C = (A, B)$, then A is a cylinder if and only if B is a cylinder.*

This proof also carries through to higher dimensions.

We come now to the proofs of the two theorems.

Proof of Theorem 1. Identify the point (i, j) in C with the monomial $x^i y^i$ and C itself with the polynomial

$$c(x, y) = \sum_{(i,j) \in C} x^i y^i .$$

Since C is a cylinder, $c(x, y)$ is the product of a polynomial in x and a polynomial in y, $c(x, y) = p(x)q(y)$. In a similar manner, associate the polynomials $a(x, y)$ with A and $b(x, y)$ with B. Since $C = (A, B)$,

$$a(x, y)b(x, y) = c(x, y) = p(x)q(y). \tag{1}$$

Now, unique factorization holds in $Z[x, y]$. So, express (1) in terms of irreducible polynomials,

$$a_1(x, y) \cdots a_n(x, y)b_1(x, y) \cdots b_m(x, y) = p_1(x) \cdots p_r(x)q_1(y) \cdots q_s(y). \tag{2}$$

Comparison of the two sides of (2) shows that each $a_i(x, y)$ and $b_j(x, y)$ is

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