ON ALGEBRAS OF HOLOMORPHIC FUNCTIONS WITH C^∞-BOUNDARY VALUES

ALEXANDER NAGEL

1. Introduction. Let $D \subset\subset \mathbf{C}^n$ be a relatively compact domain, with $\bar{D}$ its closure, and ∂D its boundary. In this paper, $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on $\mathbf{C}^n$, restricted to $\bar{D}$, $\mathcal{W}$ denotes the sheaf of germs of complex-valued Whitney C^∞-functions on $\bar{D}$, and $\mathcal{A}$ denotes the sheaf of germs of Whitney C^∞-functions on $\bar{D}$ which are holomorphic on D. For $z \in \bar{D}$, $\mathcal{O}_z$, $\mathcal{W}_z$, and $\mathcal{A}_z$ denote the stalks of the sheaves $\mathcal{O}$, $\mathcal{W}$, and $\mathcal{A}$ at z, and $\mathcal{O}(\bar{D})$, $\mathcal{W}(\bar{D})$ and $\mathcal{A}(\bar{D})$ denote the algebras of global sections of $\mathcal{O}$, $\mathcal{W}$, and $\mathcal{A}$ over $\bar{D}$.

For each $z \in \bar{D}$, there is an inclusion of rings: $\mathcal{O}_z \subset \mathcal{A}_z \subset \mathcal{W}_z$ (note that $\mathcal{O}_z \neq \mathcal{A}_z$ only if $z \in \partial D$). There is also an inclusion of algebras of global sections: $\mathcal{O}(\bar{D}) \subset \mathcal{A}(\bar{D}) \subset \mathcal{W}(\bar{D})$. In [3] it was shown that $\mathcal{W}_z$ is always a flat ring extension of $\mathcal{O}_z$, and if $z \in \partial D$, under suitable local conditions on the boundary of D near z, then $\mathcal{A}_z$ is a flat ring extension of $\mathcal{O}_z$. It was also shown that under suitable additional global conditions on the domain D, $\mathcal{A}(\bar{D})$ is a flat ring extension of $\mathcal{O}(\bar{D})$.

In this paper we continue the study of these various ring extensions, and show that, under suitable conditions, they are actually faithfully flat extensions in the algebraic sense. These results are then used to study certain finitely generated ideals in the algebra $\mathcal{A}(\bar{D})$ of holomorphic functions on D with C^∞-boundary values. More explicitly, in Section II we study the extension of local rings $\mathcal{O}_z \subset \mathcal{A}_z \subset \mathcal{W}_z$ and show:

(1) For $z \in \bar{D}$, $\mathcal{W}_z$ is always a faithfully flat ring extension of $\mathcal{O}_z$.

(2) If $z \in \partial D$, then under suitable local conditions on ∂D near z, $\mathcal{A}_z$ is a faithfully flat ring extension of $\mathcal{O}_z$.

In Section III, these local results are used to show:

(3) Under suitable global conditions on D, if $I \subset \mathcal{O}(\bar{D})$ is a finitely generated ideal, then $I \cdot \mathcal{A}(\bar{D}) = I \cdot \mathcal{W}(\bar{D}) \cap \mathcal{A}(\bar{D})$.

(4) Under suitable global conditions on D, $\mathcal{A}(\bar{D})$ is a faithfully flat ring extension of $\mathcal{O}(\bar{D})$.

The rings $\mathcal{W}(\bar{D})$ and $\mathcal{A}(\bar{D})$ have natural Fréchet topologies, and $\mathcal{A}(\bar{D})$ is in fact a closed subalgebra of $\mathcal{W}(\bar{D})$. The equality in (3) thus shows that if the larger ideal $I \cdot \mathcal{W}(\bar{D})$ is closed, then so is the smaller ideal $I \cdot \mathcal{A}(\bar{D})$. These larger ideals can be studied using methods of Malgrange [2] and Tougeron [4], and it is shown:

(5) Under suitable conditions on D, if ∂D is piecewise real analytic, and if

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