ON THE VANISHING OF THE SPAN OF A RIEMANN SURFACE

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Introduction. Lehto [5] and Ahlfors–Buerling [1] showed that if there is a point on a plane region W where every AD-function (i.e., analytic function with finite Dirichlet integral) has a vanishing derivative, then there are no nonconstant AD-functions on W. Virtanen [11] remarked that this is not true for Riemann surfaces of infinite genus, in general. Recently, Sario–Oikawa [10] posed a question whether this situation holds for HD-functions (i.e., harmonic functions with finite Dirichlet integral). Rodin [9] and Goldstein [3, 4] answered this question, but it does not seem to be final.

In this paper we shall be concerned with this question. In §1 we introduce the classes $\mathcal{S}_X^1(X = HD, KD$ or Re $AD)$ of Riemann surfaces and show the inclusion relations of them. In contrast with Lehto's results, there is a plane region which carries nonconstant HD-functions, yet there is a point on the region where every HD-function has a vanishing derivative (the derivative is taken in some fixed direction). The definition of the classes $\mathcal{S}_X$ is given in §2. W is of class $\mathcal{S}_X$ if and only if $X(W)$ does not separate the points of W. For plane regions we have $\mathcal{S}_{HD} = \mathcal{S}_{HD}^1$. §3 deals with the span of higher order. In the last section, §4, we define the sets $N_{X,s}^m(W)$ and $N_{X,a}^m(W)$, and show characterizations of the classes O_X and O_X^{2m}.

After the author had written this paper, his attention was called to papers by Lokki [6] and Myrberg [8]. In these papers they constructed the same example of a plane region which is shown in lemma 1.2.

1. The classes $\mathcal{S}_{HD}^1$, $\mathcal{S}_{KD}^1$ and $\mathcal{S}_{AD}^1$. Let W be a Riemann surface. Let $HD(W)$ be the class of harmonic functions u on W with finite Dirichlet integral $D_W(u)$, let $KD(W)$ be the class of HD-functions such that the conjugate differential du^* of du has vanishing periods along all dividing cycles, and let $AD(W)$ be the class of analytic functions on W with finite Dirichlet integral. For a point ζ of R and for a local parameter $z = x + iy$ about ζ, we define the X-span $S_X^1(\zeta, z)$ by

$$S_X^1(\zeta, z) = \left[\sup\left\{\frac{\partial u}{\partial x}(\zeta) : u \in X,\ \|u\| \equiv \sqrt{\frac{D_W(u)}{\pi}} \leq 1\right\}\right]^2$$

where X denotes the class HD, KD, or Re $AD = \{\text{Re } F \mid F \in AD\}$. We denote by $\mathcal{S}_X^1$ the class of Riemann surfaces W such that the X-span $S_X^1(\zeta, z)$

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