THE LARGEST GAPS IN THE LOWER MARKOFF SPECTRUM

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The minimum $m(f)$ of an indefinite binary quadratic form $f(x, y) = ax^2 + bxy + cy^2$ with real coefficients and positive discriminant $d(f) = b^2 - 4ac$ is defined to be the infimum of $|f(x, y)|$, taken over all pairs of integers x, y not both zero. Markoff [10] showed that for any form f, the inequality $\sqrt{d(f)}/m(f) \geq \sqrt{5}$ holds, and that there is only a countable number of possible values of $\sqrt{d(f)}/m(f)$ less than 3.

The set of all possible values of $\sqrt{d(f)}/m(f)$ is called the Markoff spectrum. It is well known that the Markoff spectrum can also be defined in terms of sequences of positive integers, as follows: Let S denote a doubly infinite sequence $\cdots, a_{-j}, \cdots, a_{-1}, a_0, a_1, \cdots, a_i, \cdots$ of positive integers and define for each integer i

$$S_i = [a_i, a_{i+1}, \cdots] + [0, a_{i-1}, a_{i-2}, \cdots]$$

(here we use the customary notation $[c_0, c_1, c_2, \cdots]$ for the simple continued fraction whose partial quotients are $c_0, c_1, c_2, \cdots$, where c_0 is an integer and the c_i, $i \geq 1$, are positive integers). Further define $M(S) = \sup S_i$, where the supremum is taken over all integers i. The Markoff spectrum is the set of all possible values of $M(S)$ as S runs through all possible doubly infinite sequences of positive integers. We shall consider the Markoff spectrum from this point of view throughout this paper.

Perron [12] was apparently the first to observe that there are gaps in the Markoff spectrum above 3. He proved that the intervals

$$(\sqrt{12}, \sqrt{13}) = (3.46410, 3.60555)$$

and

$$\left(\sqrt{13}, \frac{9\sqrt{3} + 65}{22}\right) = (3.60555, 3.66311) \tag{1}$$

contain no points of the Markoff spectrum, and that this result becomes false if either interval is enlarged.

Recently there have been a number of papers dealing wholly or partly with gaps in the Markoff spectrum above 3; for example Bumby [1], Davis and Kinney [3], Hall [6], Hightower [7], Jackson [8] and Kogonija [9].

The purpose of this paper is to give a simple method, based on continued fractions, for exactly locating gaps in the Markoff spectrum. By this we mean

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