SHELLINGS OF SPHERES AND POLYTOPES

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Introduction. Bruggesser and Mani [7] proved the boundary complex $B(P)$ of a polytope P always admits a shelling. By constructing appropriate linear functionals on certain polytopes dual to P, shellings are here produced that satisfy strong conditions on the order of appearance of P's facets. Also included are some results on shellings of more general complexes.

Our terminology is adapted from Bruggesser and Mani [7] and Grünbaum [9]. A *polytope* is the convex hull of a finite set of points and a *complex* is a finite set $\mathcal{C}$ of polytopes (in a finite-dimensional Euclidean space) such that $\mathcal{C}$ includes all faces of its members and the intersection of any two members in a face of each. The members are $\mathcal{C}$'s *cells* and the maximal members are its *facets*. Prefixes are used to indicate dimension. The $(d - 1)$-faces of a d-polytope P (the maximal members of $\mathcal{B}(P)$) are P's *facets*, and we rely on context to distinguish between the two different uses of the term *facet*. A complex is *simplicial* [resp. an *n-complex*] provided that its facets are all simplices [resp. n-cells]. The underlying set of a complex $\mathcal{C}$ is denoted by $\cup\mathcal{C}$, the notation $|\ |$ being used for cardinality.

A *semishelling* of an n-complex $\mathcal{C}$ is an ordering $(F_1, \cdots, F_m)$ of $\mathcal{C}$'s facets such that for $1 < j \leq m$ the intersection $F_j \cap (F_1 \cup F_2 \cup \cdots \cup F_{j-1})$ is either a topological $(n - 1)$-ball or the entire relative boundary of F_j; when the first alternative holds for all $j < m$ the semishelling is a *shelling*. Bruggesser and Mani [7] proved the boundary complex of a polytope always admits a shelling and any two facets can be specified as F_1 and F_m. McMullen [13] and Klee [12] used additional restrictions on the shelling order to establish other properties of polytopes; in particular, shelling was essential in McMullen's proof of the upper bound conjecture.

The main results of the present paper appear in Section 3. Theorem 3.1, which concerns the behavior of linear functionals on polytopes, may be of intrinsic interest, and it leads by means of polarity and Lemma 2.1 to the strong restrictions on shelling order contained in Theorem 3.2. These restrictions are currently being used, in the manner indicated at the end of Section 3, in a computational attack on several problems concerning the combinatorial structure of polytopes.

Section 1 contains two results on shellings of more general complexes.

1. Shellings of complexes. A complex $\mathcal{C}$ is here called a *combinatorial n-ball* provided that $\mathcal{C}$ admits a subdivision $\mathcal{S}$ combinatorially equivalent to a simpicial

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