SOLUTIONS OF DIFFERENTIAL EQUATIONS IN B-SPACES

WILLIAM J. KNIGHT

Introduction. The problem of existence of a solution for the Cauchy initial value problem

$$\text{(IVP)} \qquad x'(t) = f(t, x(t)), \quad x(0) = x_0 ,$$

when x is scalar valued, is usually attacked in one of two ways. One may assume that f has a Lipschitzian property and then obtain a unique solution by the Picard iteration method. This method is an application of the fixed point theorem for strictly contracting maps on a complete metric space. (See [10], pp. 11 and 27.) Alternatively, one may assume that f is continuous in a neighborhood $\mathfrak{N}$ of $(0, x_0)$ and construct a family of approximate solutions, among which can be found (usually by the Ascoli–Arzelà theorem) a sequence converging to a possibly non-unique solution. This is the Cauchy–Peano existence theorem. In the Carathéodory generalization of this theorem, $f(t, x)$ is required to be measurable in t for each fixed x, continuous in x for each fixed t, and bounded on $\mathfrak{N}$ by an integrable function $m(t)$. (See [5], p. 43.) In this case, a solution $x(t)$ is allowed to be absolutely continuous and satisfy (IVP) almost everywhere on some interval $J = [0, a]$.

When the functions f and x take values in a general Banach space X and the limit defining the derivative $x'(t)$ is taken with respect to the norm, (i.e., x' is a strong derivative), the Picard method for proving existence and uniqueness of a solution of (IVP) carries over with only elementary modifications. But the Cauchy–Peano existence theorem (and hence the Carathéodory generalization) fails. Dieudonné [6] and Yorke [14] have shown that when f is assumed only to be continuous in a neighborhood $\mathfrak{N}$ of $(0, x_0)$ in $[0, b] \times X$, the Cauchy problem (IVP) may fail to have a solution on any interval $[0, a]$, $0 < a \leq b$, even when X is a separable Hilbert space.

However, by putting more stringent conditions on f, it is possible to obtain existence of solutions. In 1964, Browder [2, p. 518] showed that if X is a Hilbert space, if X_w denotes the space under its weak topology, and if f is a continuous map from $[0, \infty) \times X_w$ into X_w, then the Cauchy problem (IVP) has a solution. (Although a strongly C^1 solution is claimed in [2], in fact, all that is proved is that there exists a function $x : [0, a] \to X$ such that for all $x^* \in X^*$, $x^*x(t) = x^*x_0 + \int_a^t x^*f(s, x(s))\, ds$. From this it can be shown only that x is a strong solution a.e. on $[0, a]$ and is a weak solution at every point. The author of the present paper has a counter-example to the assertion that the solution must

Received August 21, 1973.