INDICES ON C^*-ALGEBRAS THROUGH REPRESENTATIONS IN THE CALKIN ALGEBRA

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1. Introduction. Let H denote a separable, infinite-dimensional complex Hilbert space. If $\mathcal{B}$ denotes the algebra of bounded linear operators on H (with the norm topology) and $\mathcal{K}$ denotes the ideal of compact operators, the Calkin algebra $\mathcal{C}$ of H is defined to be the quotient C^*-algebra $\mathcal{B}/\mathcal{K}$. The canonical projection from $\mathcal{B}$ to $\mathcal{C}$ will be denoted by π. Whenever necessary for clarity, the Hilbert space in question will be indicated by notation like $\mathcal{B}(H)$ or $\mathcal{C}(H)$. $\mathcal{B}_n$ will denote $\mathcal{B}(H^n)$ (where $H^n = H \otimes \mathbf{C}^n$), with similar definitions for $\mathcal{K}_n$, $\mathcal{C}_n$, and π_n.

Let M_n denote the algebra of $n \times n$ complex matrices, with standard basis $\{e_{ij|i,j=1,\ldots,n}\}$. Then $\mathcal{B}_n$ is isomorphic to $\mathcal{B} \otimes M_n$. $\mathcal{K}_n$ corresponds to $\mathcal{K} \otimes M_n$ under this isomorphism, so that an isomorphism of $\mathcal{C}_n$ with $\mathcal{C} \otimes M_n$ is induced. Let $\mathcal{C} \otimes e_{ii}$ be the set $\{C \otimes e_{ii} : C \in \mathcal{C}\}$. Then $\bigoplus_{i=1}{}^n \mathcal{C} \otimes e_{ii}$ is a C^*-subalgebra of $\mathcal{C}_n$; we shall denote this subalgebra by $\bigoplus_{i=1}{}^n \mathcal{C}$.

The standard (integer-valued) index on the Fredholm operators in $\mathcal{B}$ induces an index on the invertible elements of $\mathcal{C}$. We shall use ind to denote either of these index maps, as context will clearly indicate which is meant in any given situation.

We now define indices on C^*-algebras, following the suggestion of Coburn, Douglas, Schaeffer, and Singer [11]. Let $\mathcal{A}$ be any C^*-algebra with identity. Let $\mathcal{A}^0$ denote the multiplicative group of invertible elements in $\mathcal{A}$, and let $\mathcal{A}^d$ be the discrete group consisting of $\mathcal{A}^0$ modulo its identity component. Then a topological index on $\mathcal{A}$ is defined to be an element of Hom $(\mathcal{A}^d, \mathbf{Z})$. Since the map ind on $\mathcal{C}$ induces a topological index on $\mathcal{C}$ (which is actually an isomorphism between $\mathcal{C}^d$ and the integers), we are led to define an analytic index on $\mathcal{A}$ as a map of the form ind $\circ\, \rho$, where $\rho : \mathcal{A} \to \mathcal{C}$ is an identity-preserving representation of $\mathcal{A}$ in $\mathcal{C}$. We shall denote the set of analytic indices on $\mathcal{A}$ by $A(\mathcal{A})$.

It is obvious from the standard index theory for Fredholm operators on a Hilbert space that every analytic index on $\mathcal{A}$ induces a unique topological index on $\mathcal{A}$. Thus we have a well-defined map $\alpha : A(\mathcal{A}) \to$ Hom $(\mathcal{A}^d, \mathbf{Z})$ for each $\mathcal{A}$. (Perhaps α should be written as $\alpha(\mathcal{A})$, but this notation is a bit cumbersome and does not add a great deal to clarity.) The rest of this paper is an investigation of this map α.

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