PERIODIC SOLUTIONS OF SECOND ORDER LAGRANGIAN SYSTEMS

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Introduction. Professor M. S. Berger [3], [4], [5] has introduced variational techniques for the investigation of periodic solutions of Lagrangian systems of the form $\ddot{u} + Bu + F(u, \dot{u}) = 0$, by converting the system to a nonlinear gradient operator equation in the Hilbert space of 2π periodic absolutely continuous functions with square integrable derivative. This direct Hilbert space approach is apparently unsuitable for dealing with more complex systems of the form $\ddot{u} + A\dot{u} + Bu + F(u, \dot{u}, \ddot{u}) = 0$. The linearized part yields an unwieldly operator of the form $(\lambda L_1 + \lambda^2 L_2)$ and in general it is not possible to define the nonlinear operator corresponding to $F(u, \dot{u}, \ddot{u})$. By converting the system to a gradient operator equation in the Banach space of 2π periodic continuously differentiable functions and introducing an augmented system of equations we remove these difficulties and extend Professor Berger's methods to prove the existence of one-parameter families of distinct periodic solutions of a large class of autonomous Lagrangian systems $\ddot{u} + A\dot{u} + Bu + F(u, \dot{u}, \ddot{u}) = 0$. This is accomplished by showing that if a gradient operator equation is reduced, in the standard manner, to a related operator equation in a more suitable subspace, the related operator is also a gradient operator. We include the proof of this property, original with the author, for completeness and because of its many applications.

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1. An Invariant Property of Gradient Operators. For convenience we review the definition of a gradient operator. Let X be a real Banach space and denote the space of bounded linear functionals on X by X'. For X a Hilbert space identify X' with X. If $y \in X'$ we express $y(x)$ as $y(x) = \langle x, y \rangle$. An operator T, mapping an open subset of X into X' is said to be a gradient operator if there exists a real-valued continuous and differentiable functional $\mathfrak{J}$, such that $\mathfrak{J}'(x_0)x = \langle x, T(x_0) \rangle$ for all $x \in X$ and x_0 in the domain of T. $\mathfrak{J}$ is called a potential operator (for T) [2].

Let us assume X is a Hilbert space, Y a Banach space and T a map of an open subset of $X \times Y$ into X such that $T(x, y)$ is a gradient operator in x for each fixed y. If U is a closed linear subspace of X then T can be expressed as

$$T(x, y) = T(u + u^{\perp}, y) = T^{*}(u + u^{\perp}, y) + T^{**}(u + u^{\perp}, y).$$

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