ON COMMUTANTS OF COMPACT OPERATORS

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1. Introduction. Let H be a complex Hilbert space and A a bounded operator on H. Let Lat A and $(A)'$ represent the lattice of closed invariant subspaces of A and the commutant of A respectively. P. Rosenthal and D. Sarason, independently, raised the question: If $B \in (A)'$ and Lat $A \subseteq$ Lat B, is B a weak limit of polynomials in A? In this paper we show that this holds for compact operators with spanning root vectors. We also show that such operators are in class (d.c.) as defined in ([6]).

An algebra of operators U on H is reductive if every invariant subspace of U reduces U. The *reductive algebra problem* is the question: is every weakly closed reductive algebra a Von Neumann algebra. If U is generated by a single operator A, then A is called a reductive operator. The reductive operator problem was shown to be equivalent to the invariant subspace problem ([2]). Here we consider reductive algebras, generated by a single operator A, which contain a compact operator K. We show that if 0 is not an eigenvalue of K then A is normal. The results of P. Rosenthal [4, Chap. 5] and T. Andô ([1]) are easily obtained from this theorem. In the proof, the techniques of Rosenthal and Andô are extended by means of the recent invariant subspace theorem by Lomonosov for operators commuting with compact operators [3]. We also show that if K has spanning root vectors then A is normal. An example is given to show that any weakening of these results leads to a solution of the general reductive operator problem.

2. Preliminaries. We will denote the algebra of all bounded linear operators on H by $B(H)$. The commutant and double commutant of A will be denoted by $(A)'$ and $(A)''$ respectively. For n a positive integer, $H^{(n)}$ denotes the direct sum of n copies of H, and if $A \in B(H)$, $A^{(n)}$ denotes the direct sum of n copies of A acting on $H^{(n)}$ in the standard fashion. If U is an algebra on H, then $U^{(n)} = \{A^{(n)} : A \in U\}$.

The spectrum of A will be denoted by $\sigma(A)$ and the full spectrum of A, denoted $N(\sigma(A))$, is the union of $\sigma(A)$ and all the bounded components of the resolvent of A. We will use the simple observation that if $\sigma(A)$ is countable then $N(\sigma(A)) = \sigma(A)$.

Our first lemma is a well known characterization of the weak closure of an operator algebra.

LEMMA 1. *Let U be a weakly closed sub-algebra of $B(H)$. If* Lat $U^{(n)} \subseteq$ Lat $B^{(n)}$ *for positive integers $n \geq 1$ then $B \in U$.*

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