PRIMITIVE ELEMENTS FOR MODULES OVER $\mathcal{O}(Y)$

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Let X, Y be normal Stein spaces and let $f : X \to Y$ be a proper holomorphic map. To avoid technical problems we assume that $\dim X \equiv \dim Y \equiv M$ and that f is onto. In [1] Stout considers the following question: When does there exist $H \in \mathcal{O}(X)$ such that H and a finite number of its powers generate $\mathcal{O}(X)$ as a module over $f^*\mathcal{O}(Y)$? Such an H is called a primitive element for $\mathcal{O}(X)$ over $f^*\mathcal{O}(Y)$. Stout's main result is the following:

PROPOSITION 1. *Suppose X, Y are manifolds, Y is connected, and f is a covering map. Such an H exists if the group of covering transformations is cyclic and $H_1(X, Z)$ is free abelian.*

Stout notes in [1] that difficulties arise if f is not everywhere regular. The purpose of this note is to investigate the effect of the presence of singularities of f, or the spaces X, Y, on the problem of finding primitives. In order to do this we recast the problem.

PROPOSITION 2. *The following are equivalent:*

a) *There is a primitive element H for $\mathcal{O}(X)$ over $f^*\mathcal{O}(Y)$*

b) *There is $H \in \mathcal{O}(X)$ such that the map $g : X \to Y \times \mathbf{C}$ defined by $g(p) = (f(p), H(p))$ is an embedding.*

Proof. Suppose a primitive element H exists. Define h: $Y \times \mathbf{C} \to Y$ via projection. For any compact set $K \subset Y \times \mathbf{C}$, $g^{-1}(K) = f^{-1}(h(K))$ so g is proper and $\bar{X} = g(X)$ is an analytic subset of $Y \times \mathbf{C}$. Since X is Stein the elements of $O(X)$ separate points and define local coordinates near each point of X, g must be injective and have $rkg_*(p) = \dim_{\mathbf{C}} T(X, p)$ for all $p \in X$. Thus g is an embedding.

Conversely, suppose g is an embedding. Let z be a coordinate on $\mathbf{C}$. Set $\bar{h} = h \mid \bar{X}$. Since X is Stein and f is proper f has discreet fibers and so also does $\bar{h}$. Dim $X = M$ so we see from [2], p. 127, that $rk\bar{h}_*(p) = M$ for an open dense set A of regular points $p \in \bar{X}$. As in [3], lemmas 1.6 and 1.8, one sees that $\bar{X} - A$ is in fact an analytic subset of $\bar{X}$ of dimension $\leq M - 1$. Since $\bar{h}$ is proper, $B = \bar{h}(\bar{X} - A)$ is an analytic set and $\bar{h} : \bar{X} - \bar{h}^{-1}(B) \to Y - B$ is a covering map of degree l. As in [4], thm 12, p. 102, we construct a polynomial $P(Z) = \sum_{i=0}^{l} P_i Z^i$, $P(Z) \in O(Y - B)[Z]$ such that $\sum_{i=0}^{l} P_i \circ \bar{h}z^i \equiv 0$. Because Y is normal the arguments of [4] show that the P_i extend to elements of $O(Y)$. Embed Y as a closed subvariety of $\tilde{Y} \subset \mathbf{C}^{2n+1}$. We have the following sequence of embeddings, $X \to Y \times \mathbf{C} \to \mathbf{C}^{2n+2}$. Denote the image of X in

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