ON THE EXISTENCE OF NON-METRIZABLE HEREDITARILY LINDELÖF SPACES WITH POINT-COUNTABLE BASES

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In [P] Ponomarev proved that if there is a Souslin space (i.e. a non-separable linearly ordered topological space satisfying the countable chain condition) then there is a hereditarily Lindelöf regular T_1 space with a point-countable base which is not metrizable. (See also [B].) Ponomarev asked whether such an example could be produced without assuming the existence of a Souslin space. We do not answer his question, but translate the problem into a question of the existence of a certain family of subsets of ω_1, as well as relating it to the normal Moore space problem.

THEOREM 1. *There is a hereditarily Lindelöf regular T_1 space with a point-countable base which is not metrizable if and only if there is an uncountable family $\mathcal{B}$ of subsets of ω_1 satisfying the following conditions.*

(1) *If $\mathcal{B}' \subset \mathcal{B}$ is uncountable, $\bigcap \mathcal{B}' = \emptyset$.*

(2) *For each $\mathcal{B}' \subset \mathcal{B}$, there is a countable $\mathcal{B}'' \subset \mathcal{B}'$ such that $\bigcup \mathcal{B}'' = \bigcup \mathcal{B}'$.*

(3) *For every distinct α and β in ω_1, there is $B \in \mathcal{B}$ with $\alpha \notin B$, $\beta \in B$.*

(4) *For every $\alpha \in \omega_1$, and every $B \in \mathcal{B}$ containing α, there is a $C \in \mathcal{B}$ such that $B \supset \bigcap \{\omega_1 - D : D \in \mathcal{B}, D \cap C = \emptyset\}$, and $\alpha \in C$.*

(5) *If $\alpha \in B \cap C$, $B, C \in \mathcal{B}$, there is $D \in \mathcal{B}$ such that $\alpha \in D \subset B \cap C$.*

It should be pointed out that the existence of a Souslin space is consistent with the axioms of set theory [J], [Te], [Je]. In [ST] Martin's Axiom is introduced and it is shown that Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ is consistent with the axioms of set theory and implies there is no Souslin space. In [T_4] I proved, under the assumption of Martin's Axiom plus $2^{\aleph_0} > \aleph_1$, that if a hereditarily Lindelöf regular T_1 space with a point-countable base is the intersection of fewer than $2^{\aleph_0}$ open sets in its Stone–Čech compactification, then it is metrizable. It follows from the main theorem in [F] that paracompact—a fortiori Lindelöf—regular T_1 spaces with point-countable bases which are G_δ's in their Stone–Čech compactifications are metrizable. ("Absolute G_δ's" are p-spaces.) Šapirovskiĭ [$\check{S}_2$] has shown that countable chain condition absolute G_δ's with point-countable bases are metrizable.

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