ON THE TRANSFORMATION OF LOG $\eta(\tau)$

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Let $\eta(\tau)$ be the Dedekind η-function, defined by the formula

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}), \qquad \operatorname{Im}(\tau) > 0.$$

Dedekind [1] proved the following law of transformation of log $\eta(\tau)$ under the action of the elliptic modular group: If

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}),$$

then

$$(1) \qquad \begin{aligned} \log \eta(\sigma(\tau)) &= \log \eta(\tau) + \frac{\pi i b}{12} \qquad (c = 0) \\ \log \eta(\sigma(\tau)) &= \log \eta(\tau) + \tfrac{1}{2} \log \left(\frac{c\tau + d}{i}\right) + \pi i \frac{a + d}{12c} - \pi i s(d, c) \quad (c > 0), \end{aligned}$$

where $\sigma(\tau) = (a\tau + b)/(c\tau + d)$, all logarithms are taken with respect to the principal branch, and $s(d, c)$ is a *Dedekind sum* defined by

$$(2) \qquad s(d, c) = \sum_{\mu (\mathrm{mod}\, c)} \left(\left(\frac{d\mu}{c}\right)\right)\left(\left(\frac{\mu}{c}\right)\right),$$

and where

$$(3) \qquad \begin{aligned} ((x)) &= \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer} \\ 0 & \text{otherwise} \end{cases} \\ &= -\sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n x}}{2\pi i n}. \end{aligned}$$

The proof of the first relation of (1) is trivial, but the proof of the second is fairly involved. In addition to the original argument of Dedekind, other proofs have been given by Rademacher [5, 6], Siegel [7], Iseki [4] and Weil [8]. The proofs of Siegel and Weil are only for $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The paper [6] of Rademacher contains an ingenious generalization of Siegel's argument for the case of general σ. Weil's argument is, perhaps, the shortest and at the same time, the most natural of the arguments. It deduces the transformation

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