THE NONSTANDARD HULL OF A NORMED RIESZ SPACE

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Nonstandard hulls of normed spaces have been investigated in some detail ([9], [2], [4]). In the present paper the authors consider the special case of nonstandard hulls of normed Riesz spaces.

Let (L, ρ) be a normed Riesz space and let $(\hat{L}, \hat{\rho})$ be the nonstandard hull with respect to an $\aleph_1$-saturated enlargement. It is easy to see that $(\hat{L}, \hat{\rho})$ is a Banach lattice and that L is a Riesz subspace of $\hat{L}$. Although for an arbitrary normed space (E, d), E is always contained in a hyperfinite dimensional subspace of $\hat{E}$ (i.e., in a subspace of the form $\pi(S \cap \text{fin}_d\,({}^*E))$ where S is a *-finite dimensional subspace of ${}^*E$ and π is the canonical quotient map of $\text{fin}_d\,({}^*E)$ onto $\hat{E}$), L is contained in $\pi(S \cap \text{fin}_\rho\,({}^*E))$ where S is a *- (finite dimensional Riesz) subspace of ${}^*L$ if and only if every finite subset of L can be uniformly approximated by elements of finite dimensional Riesz subspaces ((L, ρ) has the finite dimensional Riesz subspace property—Section 2). In particular (L, ρ) has this property if L has the principal projection property.

In Section 3 it is shown that a separable normed Riesz space (F, η) with the finite dimensional Riesz subspace property is norm and order isomorphically embeddable in $(\hat{L}, \hat{\rho})$ if and only if (F, η) is Riesz finitely representable in (L, ρ) (for every finite dimensional Riesz subspace S of F and $\delta > 0$ there is a Riesz isomorphism T of S into L such that $\eta(y) \leq \rho(Ty) \leq (1 + \delta)\eta(y)$ for all $y \in S$). In Section 4 it is shown that most of the usual order completeness properties are equivalent for $\hat{L}$ and that they are, in turn, equivalent to the statement that c_0 is not Riesz finitely representable in (L, ρ). Furthermore it is shown that if (L, ρ) is a Banach lattice, it is super-reflexive if and only if neither c_0 nor l_1 is Riesz finitely representable in (L, ρ).

In Section 5 the ideal $\hat{I}(L)$ generated by L in $\hat{L}$ is introduced. It is shown that again the usual order completeness properties are equivalent for $\hat{I}(L)$ and they are, in turn, equivalent to the (A, iii) property for (L, ρ). Necessary and sufficient conditions are given for L to be an ideal, a band, or a projection band in $\hat{L}$. Finally Section 6 contains a retraction theorem for positive linear functionals.

The authors wish to express their appreciation to Ward Henson for many helpful remarks in the preparation of this paper and to Peter Meyer-Nieberg for conversations about his results.

Received October 25, 1973. A portion of the material presented in this paper is part of a thesis presented as partial fulfillment of the requirements for the Ph.D. degree at Duke University.