SUBLATTICES GENERATED BY CHAINS IN MODULAR TOPOLOGICAL LATTICES

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B. Jónsson has given necessary and sufficient conditions for a subset of a modular lattice to generate a distributive sublattice [6]. In this paper we give conditions under which a sublattice generated by chains is a meet of those chains. Using these results we show that certain topological lattices are the finite union of distributive sublattices. Another consequence is that in compact connected breadth two lattices the partially ordered sets of meet irreducibles and join irreducibles have the same width.

If L is a lattice and $x \in L$, then $M(x) = \{y \in L : x \leq y\}$, and $L(x) = \{y \in L : y \leq x\}$. If $x \leq y$, then $[x, y] = M(x) \cap L(y)$. The *width* $w(X)$ of a partially ordered set X is the maximum number of elements in a set of incomparable elements. We denote the set of all meet (join) irreducible elements by $MI(L)[JI(L)]$. A topological lattice is a lattice L endowed with a Hausdorff topology for which the lattice operations $\wedge$ and $\vee$ are continuous functions from $L \times L$ into L. We denote the topological closure and interior of a set A by A^* and A^0 respectively.

1. Sublattices generated by chains. We begin this section with some algebraic observations.

LEMMA 1.1. *Let L be a modular lattice. Suppose $x_1, \cdots, x_n, y_1, \cdots, y_n \in L$ and $\{P, Q\}$ is a partition of $\{1, \cdots, n\}$ such that $x_i \leq y_i$ for all $i \in P$ and $y_j \leq x_j$ for all $j \in Q$. Then*

$$(\wedge_{i=1}^n x_i) \vee (\wedge_{i=1}^n y_i) = (\wedge_{i\in P} y_i) \wedge (\wedge_{j\in Q} x_j) \wedge [(\wedge_{i\in P} x_i) \vee (\wedge_{j\in Q} y_j)].$$

Proof. Since $\wedge_{i\in P} x_i \leq \wedge_{i\in P} y_i$, the modular law yields:

$$(1) \qquad (\wedge_{i=1}^n x_i) \vee (\wedge_{i=1}^n y_i) = (\wedge_{i\in P} y_i) \wedge [(\wedge_{i=1}^n x_i) \vee (\wedge_{j\in Q} y_j)].$$

A second application of the modular law together with $\wedge_{j\in Q} y_j \leq \wedge_{j\in Q} x_j$ yields:

$$(2) \qquad (\wedge_{i=1}^n x_i) \vee (\wedge_{j\in Q} y_j) = (\wedge_{j\in Q} x_j) \wedge [(\wedge_{i\in P} x_i) \vee (\wedge_{j\in Q} y_j)].$$

Substituting the right hand side of (2) into the right hand side of (1) completes the proof.

LEMMA 1.2. *Let $C_1, \cdots, C_n$ be chains each containing 1 in a modular lattice L. Then $\wedge_{i=1}^n C_i$ is a sublattice of L if and only if for any $x_1, \cdots, x_n \in$*

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