LOCALLY QUASI-UNIFORM SPACES WITH COUNTABLE BASES

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Introduction. Throughout this paper all spaces are T_1-spaces. In [25] James Williams introduced locally uniform spaces and showed in a straightforward manner that a regular space with a σ-locally finite base admits a compatible local uniformity with a countable base. As is shown in [5], it follows from Frink's metrization theorem [8] that a locally uniform space with a countable base must be metrizable. One significance of Williams's generalization of uniform spaces is that with his results the solution to the metrization problem follows readily from Frink's metrization theorem. Thus it is more valuable to know that every locally uniform space with a countable base is metrizable than it is to know merely that every uniform space with a countable base is metrizable. The mathematical equivalence of these two metrization theorems and of the Nagata–Smirnov metrization theorem should not be viewed as an equality.

It is known that a space is quasi-metrizable if and only if it admits a quasi-uniformity with a countable base; in light of Williams' results it is natural to ask if a space is quasi-metrizable if and only if it admits a local quasi-uniformity with a countable base. This problem has been considered by several authors, though none has considered it in its natural setting as a problem concerning quasi-uniform spaces. Indeed our first result is to show that locally quasi-uniform spaces with countable bases have been studied under the following aliases: spaces having a coconvergent open neighborhood assignment, spaces admitting an o-metric that satisfies property π, γ-spaces, Nagata first countable spaces and co-Nagata spaces [22], [21], [11] and [16]. (For the sake of brevity we choose the name γ-space.)

The various characterizations of γ-spaces implicit in the above list suggest a sequence of properties each of which is possessed by any space that can be condensed onto a γ-space. It follows from the proof of [16; Theorem 3] that the weakest of these properties implies semi-stratifiability in a regular β-space. Recently the authors showed that any space with a G_δ-diagonal that admits a compatible quasi-uniformity with the Lebesgue property can be condensed onto a quasi-metrizable space [6; Theorem 5.2]. This result suggests a refinability property, which we call σ-refinability. σ-refinability is implied by metacompactness but is a much weaker refinement property. A β-space with a G_δ-diagonal that is σ-refinable is semi-stratifiable. Consequently the following equivalences hold.

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