# SEMIGROUP RINGS AS PRÜFER RINGS

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**1. Introduction.** If $R$ is an associative ring and $S$ is a multiplicative semigroup, then *the semigroup ring of $S$ over $R$* is the set of functions $f : S \to R$ that are finitely nonzero, where addition and multiplication are defined by $(f + g)(s) = f(s) + g(s)$, $(fg)(s) = \sum_{tu=s} f(t)g(u)$, where the symbol $\sum_{tu=s}$ indicates that the sum is taken over all ordered pairs $(t, u)$ of elements of $S$ with product $s$. If $\pi$ is a property of rings, then one can ask, generally, for conditions on $R$ and $S$ under which the semigroup ring of $S$ over $R$ has property $\pi$. In this paper, we restrict our attention to the case in which $R$ is a commutative ring with identity and $S$ is a torsion-free cancellative abelian semigroup with zero, and we write the semigroup operation on $S$ as addition. We think of such semigroup rings as generalizations of polynomial rings, and indeed, each polynomial ring $R[\{X_\lambda\}_{\lambda \in \Lambda}]$ over $R$ is a semigroup ring over $R$ (take $S$ to be the weak direct sum of $|\Lambda|$ copies of the additive semigroup $Z_0$ of nonnegative integers). This point of view is responsible for our choice of notation; we follow D. G. Northcott [26; p. 128] and write $R[X; S]$ for the semigroup ring of $S$ over—the elements of $R[X; S]$ are "polynomials" $r_1X^{s_1} + r_2X^{s_2} + \cdots + r_nX^{s_n}$, where each $r_i$ is in $R$ and each $s_i$ is in $S$.

In a recent paper [13] we considered the problem of determining, for $R$ and $S$ commutative, conditions under which the semigroup ring $R[X; S]$ is a GCD-domain, a unique factorization domain (UFD), or a principal ideal domain (PID). Since it is known that $R[X; S]$ is an integral domain if and only if $R$ is an integral domain and $S$ is torsion-free and cancellative, those restrictions on $R$ and $S$ were observed in [13]. A summary of the results of [13] is the following theorem (see [13] for the terminology).

THEOREM. *Assume that $D$ is an integral domain with identity and $S$ is a nonzero torsion-free cancellative abelian semigroup with zero.*

(1) *The semigroup ring $D[X; S]$ is a* GCD-*domain if and only if $D$ is a* GCD-*domain and $S$ is a* GCD-*semigroup.*

(2) *$D[X; S]$ is a* UFD *if and only if $D$ is a* UFD, *$S$ is a semigroup with unique element factorization, and each nonzero element of $H$, the maximal subgroup of $S$, is of type* $(0, 0, 0, \cdots)$.

(3) *$D[X; S]$ is a* PID *if and only if $D$ is a field and $S$ is isomorphic to $Z_0$ or to $Z$, the additive group of integers.*

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