THE DUHAMEL PRODUCT OF ANALYTIC FUNCTIONS

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In this communication we shall investigate a certain product of analytic functions, sometimes called the Duhamel product, and which is the derivative of the Mikusinski convolution product. Let D be a domain in the plane whose closure contains the origin, and let f and g be analytic in D. We consider the product

$$(1) \qquad (f * g)(z) = \int_0^z f'(z - t)g(t)\,dt + f(0)g(z) = \frac{d}{dz}\int_0^z f(z - t)g(t)\,dt.$$

With certain restrictions this product will always be defined, e.g., if $0 \in D$ and D is star-shaped with respect to the origin.

We shall first investigate certain algebras of analytic functions where the product is given by (1) and where $0 \in D$; later we shall examine a case where $0 \in \partial D$ and we show that all solutions of a certain boundary value problem of mathematical physics, the sloping beach problem, are generated by the product (1) of one fixed solution (Isaacson's solution) with all elements of a certain subalgebra of functions which themselves are not solutions.

1. Let $0 \in D$, assume that D is star-shaped with respect to the origin, and let A be the vector space of all functions analytic in D. If $f, g \in A$, then $f * g$ as defined by (1) also belongs to A, and with this multiplication A becomes an algebra. One can use results of operational calculus [2] and analytic continuation to show that A is commutative and associative (it is actually clear from (1) that $f(z) \equiv 1$ is the identity for A, and a simple change of variable shows commutativity), but we prefer to give an embedding of A into the formal power series ring $\mathbf{C}[[Z]]$ in the indeterminate Z which immediately yields associativity and commutativity and other algebraic properties of A.

Let m and n be positive integers. Let $f(z) = z^m$ and $g(z) = z^n$. Then f and g belong to A and an easy induction shows that $(f * g)(z) = (m!\,n!/(m + n)!)z^{m+n}$. We define a map $T : A \to \mathbf{C}[[Z]]$ as follows. Let $f(z)$ have the Taylor series $f(z) = \sum_{n=0}^{\infty} (a_n/n!)z^n$ for small $|z|$ and define $(Tf)(Z) = \sum_{n=0}^{\infty} a_n Z^n$. Depending on one's definitions, Tf is the Borel transform of f or is closely related to it. In any event it follows easily that if $g \in A$ and $g(z) = \sum_{m=0}^{\infty} (b_m/m!)z^m$ for small $|z|$, then

$$(f * g)(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_n b_m}{(n + m)!} z^{n+m} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} \frac{1}{n!} z^n,$$

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