THE DETERMINATION OF CONVEX BODIES BY SOME LOCAL CONDITIONS

P. W. AITCHISON

1. Introduction. A number of characterisations of the ellipsoid among convex bodies are known which are of a "global" nature, such as if all parallel sections of a convex body K are homothetic, then K is an ellipsoid [4; p. 142]. In this paper we consider some analogous "local" problems, such as if any two parallel sections of a convex body K are homothetic when the sections are "sufficiently close" to the boundary of K, then what is K? We actually use a concept generalizing the concept of "homothetic" and "constant width", but even in this simple case we show that K need not be an ellipsoid, but its surface ∂K must be composed of finitely many pieces of ellipsoids. We also generalise a result of Blaschke [3] and Olovjanischnikoff [6] in showing that if all sections of a convex body K are centrally symmetric when the sections are "sufficiently close" to the boundary of K, then K is an ellipsoid. This result eliminates the restrictive conditions previously required.

Points and vectors are denoted $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\cdots$, and in particular $\mathbf{o}$ is the origin, $\mathbf{u}$ is always a unit vector, $\mathbf{ab}$ denotes the line segment joining $\mathbf{a}$ to $\mathbf{b}$, and $\mathbf{f}(\mathbf{u})$ denotes a vector (point) valued function of $\mathbf{u}$. The usual terminology and concepts associated with a convex body K are used, as given in [7] and [4] for instance. In particular, if a support plane to K at a point $\mathbf{p}$ has the normal vector $\mathbf{u}$ which is directed away from K, then $\mathbf{u}$ is called an *outer normal* of $\mathbf{p}$. A support plane of K is called *regular* if it has a single point in common with ∂K. The *shadow boundary of K in the direction* $\mathbf{u}$ is the set of intersections with K of all support lines of K parallel to the vector $\mathbf{u}$. The *width of K in the direction* $\mathbf{u}$ is the distance between the two support planes of K which are perpendicular to the vector $\mathbf{u}$.

We introduce the concept of equivalence of two convex bodies as a natural generalisation of "homothetic bodies" and of "bodies of constant width". Two n-dimensional convex bodies K_1 and K_2 are *equivalent* if the ratio of the width of K_1 in the direction $\mathbf{u}$ to the width of K_2 in the direction $\mathbf{u}$ is constant as $\mathbf{u}$ varies. Thus K_1 is of constant width if and only if it is equivalent to the sphere, and in general any body K is equivalent to its centrally symmetric Minkowski sum body $K - (-K)$.

Other important definitions are given at the start of Section 4 and before Lemma 2.

2. The support function. The support function H of a convex body K is defined by $H(\mathbf{v}) = \sup_{\mathbf{x} \in K} \mathbf{x} \cdot \mathbf{v}$, and if the origin $\mathbf{o}$ is in the interior of K, then

Received April 17, 1973. Revisions received October 5, 1973.