AN ASYMPTOTIC FORMULA FOR EXTENDED EULERIAN NUMBERS

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1. Introduction. Fix $\lambda > 1$. Define $d_k(n)$ by $\zeta(s)^k = \sum_{n=1}^{\infty} d_k(n)n^{-s}$, where $\zeta(s)$ is the Riemann zeta function. Note that $d_k(n)$ is a multiplicative function such that $d_k(1) = 1$ and $d_k(p^a) = \binom{a+k-1}{a}$ for p prime, $a \geq 1$. For large Re (s),

$$\frac{\lambda - 1}{\lambda - \zeta(s)} = \frac{\lambda - 1}{\lambda}\cdot\frac{1}{1 - \zeta(s)/\lambda} = \frac{\lambda - 1}{\lambda}\sum_{k=0}^{\infty} \zeta(s)^k\lambda^{-k} = \sum_{n=1}^{\infty} H(n)n^{-s},$$

where $H(n) = H(n, \lambda) = \lambda^{-1}(\lambda - 1)\sum_{k=0}^{\infty} \lambda^{-k} d_k(n)$. The numbers $H(n)$ are the *extended Eulerian numbers*; when n is square-free, $H(n)$ is an *Eulerian number*. Properties of the extended Eulerian numbers may be found in [1].

Let $\Omega = \Omega(n)$ denote (as usual) the total number of prime factors of n, e.g. $\Omega(12) = 3$. In this paper we give an asymptotic formula for $H(n)$ as $\Omega \to \infty$. This formula is then used to sharpen some estimates of Hille [2] and to produce various other estimates for $H(n)$.

Hille obtained estimates for certain sums $\sum H(n)$ and therefrom deduced an upper bound and an Ω-result for $H(n)$. He remarked that his upper bound was probably not very sharp when the number of distinct prime factors of n is large. We study the growth of $H(n)$ by estimating the series $\sum_{k=0}^{\infty} \lambda^{-k} d_k(n)$ given above. This direct approach enables us to sharpen Hille's upper bound when $\Omega(n)$ is large and also to improve his Ω-result.

We remark that $H(n)$ grows at least exponentially with Ω; in fact, $H(n) \geq \lambda^{-1}(\lambda/(\lambda - 1))^{\Omega}$. For if $n > 1$,

$$\begin{aligned} H(n) &\geq \lambda^{-1}(\lambda - 1)\sum_{k=0}^{\infty} \lambda^{-k} d_k(2^{\Omega}) = \lambda^{-1}(\lambda - 1)\sum_{k=0}^{\infty} \lambda^{-k}\binom{\Omega + k - 1}{\Omega} \\ &= \lambda^{-2}(\lambda - 1)\sum_{k=0}^{\infty}\binom{\Omega + k}{\Omega}\lambda^{-k} \\ &= \lambda^{-2}(\lambda - 1)(1 - \lambda^{-1})^{-\Omega-1} = \lambda^{-1}(\lambda/(\lambda - 1))^{\Omega}. \end{aligned}$$

2. The asymptotic formula for $H(n)$. For $x \geq 0$ and positive a_i, $1 \leq i \leq \nu$, define $f(x) = \lambda^{-x}\prod_{i=1}^{\nu}\binom{a_i + x}{a_i}$ and define

$$H(a_1, \cdots, a_\nu) = \lambda^{-2}(\lambda - 1)\sum_{k=0}^{\infty} \lambda^{-k}f(k). \tag{2.1}$$

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