DUALITY THEORY WITH SEMIBIPOLAR SPACES

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The purpose of this paper is to extend duality theory from its usual context of locally convex (l.c.) spaces to a larger class of topological vector spaces (t.v.s.).

The notion of an associated l.c. topology will be used throughout. Given a vector topology T the associated l.c. topology T_{lc} is defined to be the supremum of all l.c. topologies weaker than T. It is thus the largest l.c. topology contained in T, and the family of all absolutely convex hulls of T-neighborhoods of 0 constitutes a base for its neighborhood system of 0. A t.v.s. is called nearly barreled if each barrel is a neighborhood of 0 and nearly infra-barreled if each bornivore-barrel is a neighborhood of 0.

1. Spaces of continuous linear transformations. Given F, E t.v.s. and a vector space G of continuous linear transformations from F into E, a nonempty family β of bounded subsets of F, directed by containment, generates a vector topology on G, whose general properties, including those of the equicontinuity closure theorem and hence the Alaoglu-Bourbaki theorem, are the same as in the l.c. case [3; III, 3.1, 3.3, 4.1 and 4.3]. The assumption that F, E are l.c. only enters in theorems where F is assumed to be barreled or infra-barreled. But even here, the assumption that F is l.c. is unnecessary.

THEOREM 1.1. *In the context above, let E be* l.c.

(a) *If F is nearly barreled and β is the family of all finite subsets of F, each bounded subset of G is equicontinuous.*

(b) *If F is nearly infra-barreled and β is the family of all bounded subsets of F, each bounded subset of G is equicontinuous.*

2. Duality theory. We begin with a parallel for the bipolar theorem.

THEOREM 2.1. *Let (F, T) be a* t.v.s. *and B be a T-barrel-neighborhood of* 0. *Then B is T_{lc}-closed (hence weakly closed).*

Proof. Let p be the Minkowski gauge of B. Then p is a T-continuous seminorm on F with $B = \{p \leq 1\}$. Thus B is T_p-closed and $T_p \subset T_{lc}$. Q.E.D.

This leads to the following generalization of the Mackey-Arens theorem for arbitrary t.v.s.

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