ON A THEOREM OF ARONSZAJN AND DONAGHUE ON SINGULAR SPECTRA

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In Memory of S. E. L., III

In this paper a theorem for perturbations of rank one is generalized, leading to a sufficient condition that an embedded singular continuous spectrum dissolve under perturbation.

A theorem of Aronszajn [1] and Donaghue [3[essentially states that if the difference of two self-adjoint operators has rank one, then their singular parts are mutually singular. Although this is obviously false for higher rank [2; p. 577] a certain generalization holds if the perturbation is positive definite. We confine ourselves to bounded operators, although unbounded operators may be discussed easily by following [5].

Let $T = \int \lambda dE(\lambda)$ be bounded self-adjoint on $\mathcal{H}$, let A be bounded, let $H = T + AA^*$, and assume, as usual, that the range $A\mathcal{H}$ of A is cyclic for T and H. Let $G(z) = (T - z)^{-1}$ and $R(z) = (H - z)^{-1}$. Define

$$\delta_\epsilon(T - \lambda) = \frac{1}{\pi} \frac{\epsilon}{(T - \lambda)^2 + \epsilon^2}$$

and define $\delta_\epsilon(H - \lambda)$ similarly. The singular part of the measure $d\,\langle E(\lambda)x, x\rangle$ is then supported by the set [3; §1]

$$\{\lambda : \lim_{\epsilon \downarrow 0} \langle \delta_\epsilon(T - \lambda)x, x\rangle = \infty\}$$

and hence, because $A\mathcal{H}$ is cyclic, the singular part of T is supported by the set

$$S(T, A) = \{\lambda : \lim_{\epsilon \downarrow 0} \langle \delta_\epsilon(T - \lambda)x, x\rangle = \infty \quad \text{for some} \quad x \in A\mathcal{H}\}.$$

THEOREM. *The singular part of H is supported on the complement of the set of points λ for which there is a $\delta > 0$ such that*

$$A^*\delta_\epsilon(T - \lambda)A \geq \delta I \tag{1}$$

for all sufficiently small ϵ.

Proof. If $W = B + iC$ where $C \geq \delta I$, then W is invertible and $0 \leq -\mathrm{Im}\, W^{-1} \leq \delta^{-1}I$. Indeed, $\delta + iW$ is dissipative so [7; pp. 250–251] $||(\delta + iW - \lambda)^{-1}|| \leq \lambda^{-1}$ for $\lambda > 0$. Set $\lambda = \delta$ to obtain $||W^{-1}|| \leq \delta^{-1}$ and note that $-\mathrm{Im}\, W^{-1} = W^{-1*}CW^{-1}$.

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