FREE FACTOR GROUPS OF ONE-RELATOR GROUPS

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Let $G \cong F/R$ be a one-relator group, $R \subseteq F^p[F, F]$ for some rational prime p. Cohomological methods are applied to obtain an upper bound for the rank of a free factor group of G in terms of the rank of a cup-product, which in turn is connected to the form of the relator $r \in R$. A "normal form" for r, modulo the third term of a suitable central series, is also obtained.

Let G be a finitely generated group with one defining relation. If $x_1, \cdots, x_n$ is a set of free generators of a free group F and $G \cong F/R$, where R is the normal subgroup of F generated by $r \in F$, $r \neq 1$, we write $G = \langle x_1, \cdots, x_n; r\rangle$. We assume that a fixed epimorphism $e: F \to G$ is given with kernel R.

In this paper we consider the case $r \in F^p[F, F]$ for some rational prime p (p is not uniquely determined by r), where F^p and $[F, F]$ are the subgroups of F generated by p-th powers and by commutators respectively, and we apply the cohomology of G over $\mathbf{Z}$ or $\mathbf{Z}/p\mathbf{Z}$ to study free factor groups of G, using cohomological methods of Serre and Labute [8], [5]. Theorem 1 gives an upper bound for the rank of a free factor group of G in terms of the rank of a cup-product, and Proposition 1 connects the latter to the relator r. Theorems 2 and 3 furnish a "normal form" for the relator r. The case $r \in [F, F]$ has already been investigated by A. Steinberg [10]; we obtain his main results again, this time in a cohomological setting.

Let $\mathbf{Z}$ denote the ring of rational integers, $q = 0$ or p, p a rational prime. If $r \in [F, F]$, take $q = 0$.

Set $k = \mathbf{Z}/q\mathbf{Z}$ with G given as above, and consider k as a G-module with trivial action. Let $H^i(G, k) = H^i(G)$ denote the i-th cohomology group of G with coefficients in k. Set $F_1 = F$ and $F_{m+1} = F_m{}^q[F_m, F]$ for $m > 0$, where $[F_m, F]$ denotes the subgroup generated by commutators of elements of F_m with elements of F. Clearly $F_m \supset F_{m+1}$ for all m.

In this situation it is well-known that $H^1(G) \cong \mathrm{Hom}\,(G, k) \cong \mathrm{Hom}\,(G/G_2, k)$ (G_m defined similar to F_m). Since $r \in F_2$, $F/F_2 \cong G/G_2$; hence $H^1(F) \cong H^1(G) \cong k^n$. Also $H^1(R)^F = \{f \in H^1(R) : f(x^{-1}yx) = f(y) \text{ for all } x \in F, y \in R\}$ is equal to $\mathrm{Hom}\,(R/R^q[R, F], k)$. We now verify that $R/R^q[R, F] \cong k$ so that $H^1(R)^F \cong k$. First, $R/[R, F]$ is cyclic since all conjugates of r are congruent mod $[R, F]$. Second, set $F_1{}^* = F$ and $F_{m+1}{}^* = [F_m{}^*, F]$ so that $F_m{}^*$ is the m-th term of the lower central series of F. It is well-known [7; pp. 311–312] that $\bigcap_m F_m{}^* = 1$. Since $r \neq 1$ there exists an m such that $r \in F_m{}^*$, $r \notin F_{m+1}{}^*$. Also since $F_m{}^*/F_{m+1}{}^*$ is free abelian [7; p. 342] and since $[R, F] \subseteq [F_m{}^*, F] = F_{m+1}{}^*$, $R/[R, F]$ is infinite cyclic. Since $R^q[R, F]/[R, F]$ is generated by $r^q[R, F]$, $R/R^q[R, F] \cong k$.

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