EIGEN VALUES OF TOEPLITZ OPERATORS ON SU(2)

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1. Introduction. Let $\mathbf{R}$ be the real numbers, $\mathbf{Z}$ the integers, $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, and let $L^2(\mathbf{T})$ be the Hilbert space defined by the inner product and norm

$$\langle f \mid g \rangle = \frac{1}{2\pi} \int_{\mathbf{T}} f(\theta)\overline{g(\theta)}\, d\theta, \qquad \|f\| = \langle f \mid f \rangle^{\frac{1}{2}}.$$

For $c(\theta)$ a real continuous function on $\mathbf{T}$ the operator

$$[M(c)f](\theta) = c(\theta)f(\theta), \qquad f \in L^2(\mathbf{T}), \tag{1}$$

is bounded and self-adjoint. For $n = 0, 1, \cdots$ let $E(n)$ be the projection of $L^2(\mathbf{T})$ on the $(n + 1)$-dimensional subspace spanned by $\{e^{ik\theta}\}_{k=0}^{n}$. If

$$W(c, n) = E(n)M(c)E(n)|_{E(n)L^2(\mathbf{T})}, \tag{2}$$

then $W(c, n)$ is a self-adjoint operator on $E(n)L^2(\mathbf{T})$. Let $\{\lambda_\nu^{(n)}\}_{\nu=1}^{n+1}$ be the eigen values of $W(c, n)$. Since

$$\langle W(c, n)e^{ik\theta} \mid e^{ij\theta} \rangle = \frac{1}{2\pi} \int_{\mathbf{T}} c(\theta)e^{-i(j-k)\theta}\, d\theta = c\hat{\ }(j - k) \tag{3}$$

the $\{\lambda_\nu^{(n)}\}_{\nu=1}^{n+1}$ are also the eigen values of the Toeplitz matrix

$$[c\hat{\ }(j - k)]_{j,k=0,\cdots,n} . \tag{4}$$

For each Borel set B in $\mathbf{R}$ we define $\Lambda(c, n, B) = \{\lambda_\nu^{(n)} : \lambda_\nu^{(n)} \in B\}^{\#}/(n + 1)$ where $\{\cdot\}^{\#}$ indicates the number of elements in the set $\{\cdot\}$. The probability measure $\Lambda(c, n, dx)$ affords a description of the location of the eigen values $\{\lambda_\nu^{(n)}\}_{\nu=1}^{n+1}$. Szegö [9] showed that if $\mu(d\theta) = (1/2\pi)\, d\theta$, then

$$\Lambda(c, n, \cdot) \rightharpoonup \mu(c^{-1}[\cdot]) \qquad \text{as } n \to \infty \tag{5}$$

where $\rightharpoonup$ indicates weak convergence of measures and where $c^{-1}[B] = \{\theta \in \mathbf{T} : c(\theta) \in B\}$. Equivalently

$$\int_{-\infty}^{\infty} F(x)\Lambda(c, n, dx) \to \frac{1}{2\pi} \int_{\mathbf{T}} F[c(\theta)]\, d\theta \qquad \text{as } n \to \infty \tag{5'}$$

for every real continuous function F on $\mathbf{R}$ vanishing at $\pm\infty$. In particular if $-\infty < a < b < \infty$ and if $\mu(c^{-1}[\{a\}]) = \mu(c^{-1}[\{b\}]) = 0$, then

$$\{\lambda_\nu^{(n)} : a \le \lambda_\nu^{(n)} \le {}_B\}^{\#}/(n + 1) \to \mu(c^{-1}[I_{a,b}]) \qquad \text{as } n \to \infty \tag{6}$$

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