GRAPHS, QUASISYMMETRY AND PERMUTATIONS WITH RESTRICTED POSITIONS

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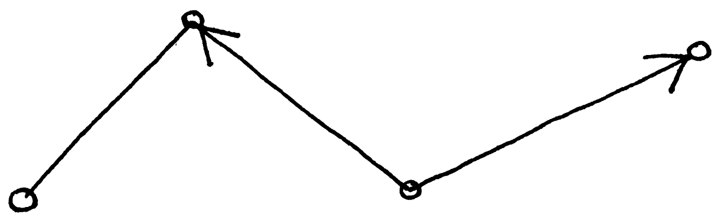
1. Introduction. In this paper we are concerned with a class of permutation problems in which the elements being permuted are restricted from certain positions. Moreover, the nature of these restrictions is that they are relative to the positions occupied by other of the elements being permuted. A typical example might be to enumerate the permutations of $\{1, 2, 3, 4\}$ in which 1 may not be adjacent to 3, 2 is not allowed to occupy the position following 4 and 4 may not be the immediate predecessor of 3. Of course, one is led to associate



the labelled diagraph with these restrictions and to note that the given problem is identical to enumerating paths of length 3 in the complement of this graph. One might also consider the related problem of enumerating 4-cycles in the complement of this diagraph.

A well-known permutation problem of this type is the n-kings problem, first solved by Wolfowitz [7] and Kaplansky [4]. Here one is asked for the number of permutations $(a_1, a_2, \cdots, a_n)$ of $Z_n = \{1, 2, \cdots, n\}$ in which $|a_{i+1} - a_i| > 1$. The associated graph is seen to be the n-path P_n with edges $(i, i+1)$ and $(i+1, i)$, $(i = 1(1)n-1)$. In his solution of the n-kings problem Kaplansky makes use of the principle of inclusion and exclusion and the notion of quasisymmetry. To be explicit, suppose that we are given events $A_1, A_2, \cdots, A_n$ and let $p(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$ denote the probability of the joint occurrence of the events $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$. We say that the given events are *quasisymmetric* if for each k and each selection $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of k events $p(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$ is either zero or a function φ_k of k alone. Thus if the events are quasisymmetric and P_0 is the probability that none of the events occur, then according to the principle of inclusion and exclusion, we obtain

$$(1.1)\qquad \begin{aligned} P_0 &= 1 - \sum_{i=1}^{n} p(A_i) + \sum_{i=j} p(A_i, A_j) - \sum_{i<j<k} p(A_i, A_j, A_k) + \cdots \\ &= 1 - c_1\varphi_1 + c_2\varphi_2 - c_3\varphi_3 + \cdots , \end{aligned}$$

Received June 28, 1973.