A CONTINUED FRACTION

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Let $(a)_r = (1-a)(1-ax)\cdots(1-ax^{r-1})$, $(x)_r = (1-x)(1-x^2)\cdots(1-x^r)$, and

$$\begin{bmatrix} r \\ s \end{bmatrix} = \frac{(x)_r}{(x)_s(x)_{r-s}}.$$

We will show that

$$(1) \qquad 1+a+b+\frac{cx-a}{1+a+bx+}\cdots\frac{cx^n-a}{1+a+bx^n} = \frac{P_n(a,b,c,x)}{Q_n(a,b,c,x)},$$

where

$$P_n(a,b,c,x) = \sum_{r=0}^{n+1} x^{\frac{1}{2}(r^2-r)} \sum_{s=0}^{\min(r,n+1-r)} b^{r-s}c^s x^{\frac{1}{2}(s^2+s)} \begin{bmatrix} r \\ s \end{bmatrix} \cdot \sum_{t=0}^{n+1-r-s} a^t \begin{bmatrix} r+t \\ r \end{bmatrix} \begin{bmatrix} n+1-s-t \\ r \end{bmatrix},$$

$$Q_n(a,b,c,x) = \sum_{r=0}^{n} x^{\frac{1}{2}(r^2+r)} \sum_{s=0}^{\min(r,n-r)} b^{r-s}c^s x^{\frac{1}{2}(s^2+s)} \begin{bmatrix} r \\ s \end{bmatrix} \cdot \sum_{t=0}^{n-r-s} a^t \begin{bmatrix} r+t \\ r \end{bmatrix} \begin{bmatrix} n-s-t \\ r \end{bmatrix}.$$

If we take $|x| < 1$, $|a| < 1$, let $n \to \infty$, and make use of the well-known identities

$$\sum_{s=0}^{\infty} b^{r-s}c^s x^{\frac{1}{2}(s^2+s)} \begin{bmatrix} r \\ s \end{bmatrix} = (b+cx)(b+cx^2)\cdots(b+cx^r)$$

and

$$\sum_{t=0}^{\infty} a^t \begin{bmatrix} r+t \\ r \end{bmatrix} = \frac{1}{(a)_{r+1}},$$

we obtain

$$(2) \qquad 1+a+b+\frac{cx-a}{1+a+bx+}\,\frac{cx^2-a}{1+a+bx^2+}\cdots = \frac{P_\infty(a,b,c,x)}{Q_\infty(a,b,c,x)},$$

where

$$P_\infty(a,b,c,x) = \sum_{r=0}^{\infty} x^{\frac{1}{2}(r^2-r)} \sum_{s=0}^{r} b^{r-s}c^s x^{\frac{1}{2}(s^2+s)} \begin{bmatrix} r \\ s \end{bmatrix} \sum_{t=0}^{\infty} a^t \begin{bmatrix} r+t \\ r \end{bmatrix} \frac{1}{(x)_r}$$
$$= \sum_{r=0}^{\infty} \frac{x^{\frac{1}{2}(r^2-r)}(b+cx)(b+cx^2)\cdots(b+cx^r)}{(x)_r(a)_{r+1}},$$

Received May 29, 1973. Revisions received September 25, 1973.