ON A THEOREM OF TORD HALL

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The theorem referred to in the title is the following.

THEOREM A. [1] *Suppose that* $u(z)$ *is subharmonic* (s.h.) *in the half plane* P *given by* $x > 0$, *where* $z = x + iy$, *that* $u(z) \leq 0$ *in* P, *and that*

$$\inf_{|z|=r} u(z) \leq -M, \qquad 0 < r < \infty. \tag{1}$$

Then

$$u(z) \leq -MA_0\left(\frac{1}{2} - \frac{|\theta|}{\pi}\right) \tag{2}$$

if $|\arg z| = |\theta| < \pi/2$, *where* $A_0 = 4\pi/(\pi^2 + 8) = .705 \cdots$.

It is tempting to conjecture that this result might hold with $A_0 = 1$. The corresponding inequality would be sharp as is shown, for instance, by

$$u(re^{i\theta}) = \frac{M(2\theta - \pi)}{2\pi}, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

which satisfies the hypotheses of Theorem A and gives equality in (2) with $A_0 = 1$ for $\theta > 0$.

We show nevertheless that such a conjecture would be false by means of the following.

Example. There exists a function $u(z)$ satisfying the hypotheses of Theorem A and such that $u(x) > -\frac{1}{2}M$ for all positive x.

We choose $c = 90$ and $a = 1/100$ so that

$$\tfrac{1}{3}c \log c > 1/a > c. \tag{3}$$

Let ϵ be a small positive number and choose $\eta = c\epsilon$, $\zeta = ie^{-i\eta}$, and set $z = re^{i\theta} = x + iy$, where $|\theta| < \pi/2$ and

$$\begin{aligned} &u(z) \\ &= -\frac{M}{\pi}\left\{\left(\theta + \frac{\pi}{2}\right) + a \log\left|\frac{z + \bar{\zeta}}{z - \zeta}\right| - \tan^{-1}\left(\frac{y - 1 + \epsilon}{x}\right) - \tan^{-1}\left(\frac{1 + \epsilon - y}{x}\right)\right\} \\ &= -\frac{M}{\pi}\left\{\theta + a \log\left|\frac{z + \bar{\zeta}}{z - \zeta}\right| + \tan^{-1}\left(\frac{|z - i|^2 - \epsilon^2}{2\epsilon x}\right)\right\}. \end{aligned}$$

Then $u(z)$ has on the imaginary axis boundary values $-M$ for $y > 0$ except for $|y - 1| \leq \epsilon$ and boundary values 0 for $y < 0$ and for $|y - 1| < \epsilon$. Also

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