ASSOCIATED PRIMES OF PRINCIPAL IDEALS

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If A is an ideal of the commutative ring R, then there are several ways of associating prime ideals of R to A. There are compelling reasons, however, why the "weakly" associated primes of Bourbaki are the most natural and in accord with Lazard [10; p. 92], "Cela (weakly) nous semble inutile, car, dans le cas noetherian, elles redonnent les notion classiques, et, dans le cas général, les notion classiques ont trés peu d'interêt." we shall call these primes the associated primes of A. Thus, a prime ideal P of R is said to be an *associated prime* of A if there exists $b \in R \backslash A$ with P a minimal prime of $A : bR$. This paper is concerned with the associated primes of principal ideals of R and of the polynomial ring $R[X]$. We shall prove that associated primes of regular elements, that is, nonzero-divisors, are well-behaved; for an integral domain D, using the representation $D = \bigcap \{D_{P_\alpha} \mid P_\alpha$ is an associated prime of a principal ideal of $D\}$, we prove a theorem on the finiteness of the ideal transform. It is also shown that the associated primes of regular elements of $R[X]$ are closely tied to associated primes of regular elements of R. This enables us to prove the stability under polynomial extension of a noetherian-like property of integral domains which is then used to obtain a result about "locally polynomial rings".

R will always denote a commutative unitary ring. A prime ideal P of R is called a *maximal prime* of the ideal A if P is maximal within the zero-divisors on R/A [9; p. 3]. A prime ideal Q of R is called a *prime divisor* of A if there exists a multiplicative system S in R such that QR_S is a maximal prime of AR_S. The maximal primes of A are precisely the maximal elements of the set of prime divisors of A [12; p. 19].

Our notation will be essentially as in [9].

We list here some facts we shall need and begin with perhaps the most useful fact about associated primes. They behave well under localization.

LEMMA 1 [11; p. 17, Proposition 5]. *Let A be an ideal of R with P a prime ideal of R. Assume that S is a multiplicative system in R and that $P \cap S = \varnothing$. If P is an associated prime of A, PR_S is an associated prime of AR_S. Conversely, if PR_P is an associated prime of AR_P, then P is an associated prime of A.*

It can happen that P is a maximal prime of an ideal A but PR_P does not consist of zero-divisors on AR_P [14], [3], and thus PR_P is not a maximal prime of AR_P. However, if we assume that an ideal has only finitely many associated primes, then the prime divisors and the associated primes coincide.

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