TRANSFORMATION OF ARITHMETIC FUNCTIONS

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1. Introduction. Let S denote the set of all arithmetic functions $f = f(n)$ for which $f(1) = 1$. Define for $n = 1, 2, \cdots$

$$\bar{f}(1) = 1;\ \bar{f}(n) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} \sum_{\substack{a_1\cdots a_r = n \\ a_i \neq 1 (i=1,\cdots,r)}} f(a_1) \cdots f(a_r),\ n > 1; \tag{1.1}$$

$$\tilde{f}(1) = 1;\ \tilde{f}(n) = \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{\substack{a_1\cdots a_r = n \\ a_i \neq 1 (i=1,\cdots,r)}} f(a_1) \cdots f(a_r),\ n > 1, \tag{1.2}$$

where each of the inner summations is over all the sets $(a_1, \cdots, a_r)$ satisfying the stated conditions.

Let β and γ denote respectively the transformations on S defined by

$$\beta(f) = \bar{f} \quad \text{and} \quad \gamma(f) = \tilde{f}, \qquad f \in S.$$

In this paper we study the properties of these transformations and consider some applications. We show that β is a bijective mapping on S, that is, β is a one-to-one mapping of S onto itself, and that its inverse mapping is γ. Equivalently,

$$\tilde{\bar{f}} = \bar{\tilde{f}} = f, \qquad f \in S. \tag{1.3}$$

We also show that if $f \circ g$ denotes the Dirichlet product of f and g, then

$$\overline{f \circ g} = \bar{f} + \bar{g},\ i.e.,\ \beta(f \circ g) = \beta f + \beta g, \tag{1.4}$$

and

$$\widetilde{f + g} = \tilde{f} \circ \tilde{g},\ i.e.,\ \gamma(f + g) = \gamma f \circ \gamma g. \tag{1.5}$$

We apply these results to solve the equation

$$f^{(k)} = g \tag{1.6}$$

$f, g \in S$ being given, and $f^{(k)} = f \circ f \circ \cdots \circ f$ (k times). We show that there are k distinct solutions which are readily and elegantly given by

$$f = \exp(2\pi is/k) \cdot \widetilde{(1/k)\bar{g}}, \qquad s = 0, 1, \cdots, k-1. \tag{1.7}$$

We also briefly consider the application of our results to the solution of the equation $f^{(k)} = fg$, $g \in S$ being given.

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