TOPOLOGICAL SEMIGROUPS, SETS OF GENERATORS, AND CONTROLLABILITY

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In this paper we define the semidirect product of two topological semigroups T and K, where the resulting product $T \times_\eta K$ is a topological semigroup. We obtain sufficient conditions for a subset $R \times J$ of the topological semigroup $T \times_\eta K$ to be a set of generators of $T \times_\eta K$. We then use our results to determine the structure of the reachable set for a class of bilinear control systems.

Introduction. Finite groups are commonly described by specifying a set of generators and relations. This gives a compact description of the group. For an arbitrary semigroup the task of displaying a set of generators and relations which completely describe the semigroup is formidable. In this paper we restrict our attention to a specific class of semigroups and derive sufficient conditions for a semigroup in this class to be described by a set of generators and relations. In [4] the concept of the semidirect product of two topological groups is examined. In this paper we define the semidirect product of two topological semigroups T and K, where the resulting product $T \times_\eta K$ is a topological semigroup, and we show that our definition is a generalization of the one for topological groups. We then consider the following question. Given a topological semigroup and given $R \subseteq T$ and $J \subseteq K$, when is $R \times J$ a set of generators of $S = T \times_\eta K$?

In Section 1 we define the semidirect product of two topological semigroups and show that the existence of a set of generators for a semigroup of the form $S = T \times_\eta K$ places certain restrictions on the structure of S. A technical result for semigroups which are embeddable in topological groups is also established. In Section 2 we find sufficient conditions for a subset $R \times J$ of a topological semigroup $T \times_\eta K$ to be a set of generators for a dense subsemigroup of $T \times_\eta K$.

In Section 3 we further restrict the class of semigroups under consideration to those embeddable in a Lie group. We find sufficient conditions for a subset $R \times J$ to be a set of generators of the whole semigroup $T \times_\eta K$. In Section 4 we discuss the controllability properties of systems described by evolution equations of the form $\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t)B_i x(t)$, where the state space

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