ON CENTRALLY SPLITTING

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Let $\mathfrak{T}$ be a torsion-torsionfree (TTF) class, in the sense of Jans [5], in the category of left modules over a ring R. Then $\mathfrak{T}$ is both a torsion class for some torsionfree class $\mathfrak{F}$, and $\mathfrak{T}$ is a torsionfree class for a torsion class $\mathcal{C}$. After Kurata [6], we shall speak of the TTF *theory* $(\mathcal{C}, \mathfrak{T}, \mathfrak{F})$ in this situation.

In this note we are interested in when the TTF theory $(\mathcal{C}, \mathfrak{T}, \mathfrak{F})$ is centrally splitting. Several equivalent conditions for centrally splitting are known [3]; three in particular are that $\mathcal{C} = \mathfrak{F}$, that the $\mathcal{C}$-torsion submodule R_c of R is generated by a central idempotent, and that R is the direct sum of its two torsion submodules, $R = R_c \oplus R_t$. We present here two more equivalences for centrally splitting; the second result (Theorem 3) is for an arbitrary hereditary torsion class and so is somewhat stronger than usual. Sandwiched between these facts is an investigation of the question: If the $\mathfrak{T}$-torsion submodule R_t of R is generated by a central idempotent, then is $(\mathcal{C}, \mathfrak{T}, \mathfrak{F})$ centrally splitting? The answer is no, in general; but it is yes if $\mathfrak{T}$ is the smallest torsion class containing R_t.

In the category ${}_R\mathfrak{M}$ of left modules over a ring R, a class $\mathfrak{T}$ is called a *torsion class* provided $\mathfrak{T}$ is closed under homomorphic images, extensions, and direct sums. We call a torsion class *hereditary* if it is also closed under submodules and *stable* if it is closed under injective envelopes. The torsionfree class $\mathfrak{F}$ associated with the torsion class $\mathfrak{T}$ is $\mathfrak{F} = \{M \in {}_R\mathfrak{M} \mid \mathrm{Hom}\,(T, M) = 0 \text{ for all } T \in \mathfrak{T}\}$. The torsion class $\mathfrak{T}$ is hereditary if and only if $\mathfrak{F}$ is closed under injective envelopes, and the hereditary torsion class is a TTF class if and only if it is closed under direct products. These definitions and results are due to Dickson [4] and Jans [5]. We prefer to refer the uninitiated reader to these sources and to the paper [3] for the fundamental facts on torsion and TTF theories rather than to try to condense that material here.

PROPOSITION 1. *Let $(\mathcal{C}, \mathfrak{T}, \mathfrak{F})$ be a* TTF *theory. Then $\mathfrak{T}$ is centrally splitting if and only if $\mathcal{C}$ is closed under essential extensions and R_t is a (module) direct summand of R.*

Proof. We need only consider the case where $\mathcal{C}$ is closed under essential extensions and $R = R_t \oplus R'$. Then $R/R' \in \mathfrak{T}$ so that $R_c \leq R'$. But $R' \in \mathfrak{F}$; hence, if $0 \neq x \in R'$, then $0 \neq R_c x \leq R_c$. Thus $Rx \cap R_c \neq 0$ so that R' is an essential extension of R_c. Thus $R_c = R'$ so that $\mathfrak{T}$ is centrally splitting.

Let $(\mathcal{C}, \mathfrak{T}, \mathfrak{F})$ be a TTF theory. If R_c is simply a module direct summand of R, then $\mathfrak{T}$ need not be centrally splitting. An example may be constructed in the ring $R = UT_2(K)$ of 2 by 2 upper triangular matrices over a field K. Let

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