SOME NEW INVERSE SERIES RELATIONS

H. W. GOULD and L. C. HSU

1. Introduction. In a series of papers [4]–[7] and [10] one of us has developed and extended the pair of reciprocal formulas (inverse series relations)

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+bk}{n} g(k) \tag{1.1}$$

and

$$\binom{a+bn}{n} g(n) = \sum_{k=0}^{n} (-1)^k \frac{a+bk-k}{a+bn-k} \binom{a+bn-k}{n-k} f(k) \tag{1.2}$$

together with the infinite series transform implied by (1.1)

$$\sum_{k=0}^{\infty} \binom{a+bk}{k} z^k g(k) = x^a \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-1}{x}\right)^n f(n) \tag{1.3}$$

where $z = (x-1)/x^b$. Here it was also assumed that $g(0) = 1$.

It is our purpose now to extend these results in a new way by replacing a and b respectively by two arbitrary sequences of numbers $\{a_i\}$ and $\{b_i\}$ in the following manner.

THEOREM 1. *Let $\{a_i\}$ and $\{b_i\}$ be two sequences of numbers such that*

$$\psi(x, n) = \prod_{i=1}^{n} (a_i + b_i x) \neq 0 \tag{1.4}$$

for all nonnegative integers x, n, with $\psi(x, 0) = 1$. Then we have the following pair of reciprocal formulas

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(k, n) g(k) \tag{1.5}$$

and

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a_{k+1} + kb_{k+1}) \psi(n, k+1)^{-1} f(k). \tag{1.6}$$

The reciprocal pair (1.1)–(1.2) follows at once when we choose $a_i = a - i + 1$ and $b_i = b$; so Theorem 1 contains an infinite variety of pairs such as this.

Theorem 1 was communicated by one of us (Hsu) in a personal letter (to Gould) in 1965 but otherwise has not been published before. We have announced this theorem in an abstract [9] and in the present paper we offer proofs of this

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