CONFORMAL MAPPINGS OF DOMAINS BOUNDED BY QUASICONFORMAL CIRCLES

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1. Introduction. A Jordan curve Γ in $\bar{\mathbf{C}}$ is called a quasiconformal circle if it is the image of a circle by a homeomorphism f which is quasiconformal in a neighborhood of that circle. A necessary and sufficient metric condition that Γ be a quasiconformal circle was given by Ahlfors [1]. We designate the chordal distance between the points $z_1, z_2 \in \bar{\mathbf{C}}$ by $q(z_1, z_2)$ and define the chordal cross ratio of the quadruple z_1, z_2, z_3, z_4 in $\bar{\mathbf{C}}$ by

$$X(z_1, z_2, z_3, z_4) = \frac{q(z_1, z_2)q(z_3, z_4)}{q(z_1, z_3)q(z_2, z_4)}.$$

Ahlfors' condition is that Γ is a quasiconformal circle if and only if

$$\sup \{X(z_1, z_2, z_3, z_4) + X(z_2, z_3, z_4, z_1)\}$$

is finite, where the supremum is taken over all ordered quadruples on Γ. The following definition is motivated by this condition.

DEFINITION 1.1. A Jordan curve Γ is a *k-circle*, where $0 < k \leq 1$, if for all quadruples of points on Γ,

$$X(z_1, z_2, z_3, z_4) + X(z_2, z_3, z_4, z_1) \leq \frac{1}{k}. \tag{1.1}$$

For completeness we say an arbitrary Jordan curve is a 0-circle.

Since the chordal cross ratio is invariant under Möbius transformations, it is easily verified that a 1-circle is a Euclidean circle or straight line. Thus the class of k-circles interpolates, as k runs from zero to one, between arbitrary Jordan curves and the simplest Jordan curves.

In the following we will be concerned with conformal maps of the unit disk onto domains bounded by k-circles. We will call such a domain a *k-domain*. Simple but laborious calculation verifies that the domain $D_k = \{z : |\arg (z)| < \pi - \arcsin k\}$ is a k-domain, and we will use this domain as an example in much of the following. In Section 2 we derive a Koebe theorem for conformal maps onto k-domains, and in Section 3 we consider distortion theorems and an estimate on the second coefficient.

We will use the following notation. $B(z_0, r)$ is the open disk $\{z : |z - z_0| < r\}$, $B(r) = B(0, r)$ and $B = B(1)$. $S(z_0, r)$ is the circle $B(z_0, r)$, $S(r) = S(0, r)$

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