POSITIVE QUASI-ORDERS ON SEMIGROUPS

MOHAN S. PUTCHA

Introduction. Positive quasi-orders on semigroups have been studied from different points of view by Tamura [10], [12], [13] and the author [5]. In the present paper we study yet other aspects of positive quasi-orders on semigroups. We relate them to the study of subdirect products of nil semigroups and of subsemigroups of $\mathfrak{N}$-semigroups. Some preliminary results on when a positive mapping attains a maximum are also obtained.

1. Preliminaries. Throughout, S will denote a semigroup and Z^+ the set of positive integers. If $a, b \in S$, we say $a \mid b$ (a divides b) if $b \in S^1aS^1$. A semigroup S is called archimedean if for any $a, b \in S$, $a \mid b^i$ for some $i \in Z^+$. We will also need some concepts and results on semilattice decompositions of semigroups and $\mathcal{S}$-indecomposable semigroups (see for example Tamura [8], [9], [10], [11] or the author [3], [4]). Finally we will need the notion of a partially ordered semigroup [1; p. 153].

By a quasi-order is meant a reflexive and transitive relation. A quasi-order π on S is called *positive* if for any $a, b \in S$, $a\pi ab$ and $a\pi ba$. This is of course the same as saying $\mid \subseteq \pi$. Let $(P, \leq)$ be a partially ordered set. A mapping $\varphi : S \rightarrow P$ is called *positive* if $\varphi(ab) \geq \varphi(a)$ and $\varphi(ab) \geq \varphi(b)$ for all $a, b \in S$. There is of course a natural correspondence between positive quasi-orders and positive mappings [5]. This correspondence is so obvious that we use it without further comment.

DEFINITION. Let φ be a mapping of S into a partially ordered set $(P, \leq)$. Correspondingly, let π be a quasi-order on S.

(1) An element $u \in S$ is a φ-idempotent if $\varphi(u^i) = \varphi(u)$ for all $i \in Z^+$. $b \in S$ is called φ-periodic if a power of b is a φ-idempotent. S is φ-periodic if each element of S is φ-periodic.

(2) An element $u \in S$ is a π-idempotent if $u^i \pi u \pi u^i$ for all $i \in Z^+$. (If π is positive, this is the same as saying $u^i \pi u$ for all $i \in Z^+$.) $b \in S$ is called π-periodic if a power of b is a π-idempotent. S is π-periodic if each element of S is π-periodic.

(3) S is φ-archimedean if for all $a, b \in S$ there exists $i \in Z^+$ such that $\varphi(a) \leq \varphi(b^i)$.

(4) S is π-archimedean if for all $a, b \in S$ there exists $i \in Z^+$ such that $a \pi b^i$.

2. Subdirect product of nil semigroups. For notions of subdirect products of semigroups see, for example, [7]. Schein [7] has remarked that a semigroup

Received March 24, 1973. The author was supported by a National Science Foundation Graduate Fellowship.