A NOTE ON LOCAL SOLVABILITY

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Introduction. As is well known the differential operator $D_x + ix^k D_y$ is not locally solvable at the origin when k is an odd integer, whereas when k is an even integer it is both solvable and analytic-hypoelliptic [3], [5]. This example led to fundamental work on the question of local solvability for principal type operators [1], [4].

In the following we consider the above example when the condition that k be a nonnegative integer is relaxed. Of course x^k is not smooth anymore and restrictions on allowable solutions must be imposed so the multiplication is well defined. This permits the elementary analysis below.

We set $P(\lambda) = D_x + ix^\lambda D_y$ where x^λ is defined as $|x|^\lambda$ if $x > 0$ and $|x|^\lambda e^{i\pi\lambda}$ if $x < 0$. Thus x^λ is locally integrable only for Re $\lambda > -1$ and would need further definition at the origin to define a distribution for Re $\lambda < -1$, e.g., $(x + i0)^\lambda$ in Gel'fand and Shilov. Fortunately we are able to avoid this complication in the following.

Proposition 1. *$P(\lambda)u = f$ is not locally solvable at the origin if λ is real, $\lambda > -1$, $\cos \pi\lambda < 0$, and u is required to be an element of $(C_0^\lambda(\Omega_x) \hat{\otimes} C_0^\infty(\Omega_y))'$.*

Of course f is an element of $C_0^\infty(\Omega)$ where we have taken $\Omega = \Omega_x X \Omega_y$ as a rectangular open neighborhood of the origin. We now explain the other notation. $C_0^\lambda(\Omega_x)$ for Ω_x an open interval of the real line is the inductive limit of the Banach spaces $C_0^\lambda(K_x)$ which are in turn the completions of $C_0^\infty(K_x)$ with norm $||u||_\lambda = \sup (|D^{[\lambda]}u(x) - D^{[\lambda]}u(x')|/|x - x'|^{\lambda-[\lambda]})$. The sup is over all pairs (x, x') in $K_x X K_x$, and as usual $[\lambda]$ is the integral part of λ. Since we are allowing $-1 < \lambda < 0$ we interpret $D^{-1}u$ as any primitive of u.

Thus $C_0^\lambda(K_x) \hat{\otimes} C_0^\infty(K_y)$ has seminorms $\sup_y ||D_y{}^j u(x, y)||_\lambda$ for each j and $C_0^\lambda(\Omega_x) \hat{\otimes} C_0^\infty(\Omega_y)$ is the inductive limit of these spaces.

Before proving the proposition we remark that the standard result is nonsolvability for $\lambda \geq 0$, $\cos \pi\lambda = -1$, and u an element of $(C_0^\infty(\Omega_x) \hat{\otimes} C_0^\infty(\Omega_y))'$, i.e., a distribution in Ω.

Proof. The usual estimate violation will be employed [2] which very roughly attempts to find a nice sequence v_j in ker ${}^tP$ such that $v_j \to \delta$. This then gives a contradiction to the solvability of $Pu = f$ by $0 = \langle u, {}^tPv_j\rangle = \langle f, v_j\rangle \to f(0)$.

Explicitly we first assume $P(\lambda)u = f$ is solvable in an open rectangular neighborhood $\Omega = \Omega_x X \Omega_y$ of the origin, i.e., for each f in $C_0^\infty(\Omega)$ there exists a solution u in $(C_0^\lambda(\Omega_x) \hat{\otimes} C_0^\infty(\Omega_y))'$. Next we pick $\omega = \omega_x X \omega_y$, another open rectangular neighborhood of the origin, such that $\omega \Subset \Omega$.

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