ON THE EXISTENCE OF NONTRIVIAL RECURRENT MOTIONS

ROBERT J. SACKER AND GEORGE R. SELL

1. Introduction. Consider a flow $\pi : X \times R \rightarrow X$ on a compact connected metric space X and let $\mathfrak{M} = \mathfrak{M}(X)$ denote the collection of all minimal closed invariant subsets of X. Let $\mathfrak{I} \subseteq \mathfrak{M}$ be a distinguished subcollection of minimal sets. We wish to find conditions on the flow π which imply that $\mathfrak{I} \neq \mathfrak{M}$. For example if $\mathfrak{I}$ is the collection of all stationary points of π, then under what conditions does there exist a nontrivial recurrent motion, i.e., a minimal set that is not a stationary point? In general let us refer to those minimal sets in $\mathfrak{I}$ as being *trivial* and those, if any, in $\mathfrak{M} - \mathfrak{I}$ as *nontrivial.*

Recall that in the case X is a compact 2-dimensional manifold, the manner in which stationary points are joined by transit orbits plays a crucial role in determining the existence of periodic orbits [3]. An orbit $\gamma(x) = \{\pi(x, t) : t \in R\}$ is a transit orbit if the limit sets α_x and ω_x are stationary points P and Q.

In the above case the stationary points P and Q are examples of two minimal sets that are l-related, a concept which we now define for a general flow. We shall say that two minimal sets P and Q are *l-related* (limit set related) if there is an x in X such that $\alpha_x \cap P \neq \emptyset$ and $\omega_x \cap Q \neq \emptyset$ or vice versa. We use this to define a relation $\sim$ on $\mathfrak{I}$. Let P and Q be two minimal sets in $\mathfrak{I}$. We shall say that $P \sim Q$ if there exist minimal sets $\{M_0, \cdots, M_k\}$ in $\mathfrak{I}$ such that $P = M_0$, $Q = M_k$ and M_{i-1} and M_i are l-related. Clearly $\sim$ is an equivalence relation on $\mathfrak{I}$. Let $\{\mathfrak{I}\} = \{\mathcal{P}_1, \mathcal{P}_2, \cdots\}$ be the partitioning of $\mathfrak{I}$ into equivalence classes under $\sim$, i.e., two minimal sets P and Q belong to the same equivalence class $\mathcal{P}_i$ if and only if $P \sim Q$. Note that if $\mathfrak{I}$ contains a finite number of minimal sets, then card $\{\mathfrak{I}\}$, the number of equivalence classes in $\{\mathfrak{I}\}$, is finite.

We will prove the following theorem.

THEOREM 1. *Assume that $\mathfrak{I}$ is a finite subcollection of $\mathfrak{M}$. If* card $\{\mathfrak{I}\} \geq 2$, *then there is a nontrivial minimal set in $\mathfrak{M} - \mathfrak{I}$.*

This is clearly equivalent to the following theorem.

THEOREM 2. *Let K be an invariant continuum in X and let $\mathfrak{M}(K)$ be the collection of all minimal closed invariant subsets of K, and assume that $\mathfrak{M}(K) = \{M_1, \cdots, M_\sigma\}$ is finite. Then for all i, j one has $M_i \sim M_j$.*

We have already seen one interpretation of Theorem 1 in the case that $\mathfrak{I}$ is the collection of all stationary points for the flow π. Another interpretation

Received December 11, 1972. Revisions received May 24, 1973. The first author was supported in part by the United States Army under Contract DA-ARO-D-31-124-71-G14. The second author was supported in part by NSF Grants No. 27275 and GP-38955.