FERMAT'S CONJECTURE, ROTH'S THEOREM, PYTHAGOREAN TRIANGLES, AND PELL'S EQUATION

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It is shown that for fixed $v \geq 1$ and prime $p \geq 3$ there are at most a finite number of relatively prime, positive integer pairs (x, y) on the line $y = x + v$, for which $x^p + y^p$ is a p-th power. This is a consequence of Roth's theorem, stating that a real algebraic irrational is approximable to no order higher than 2. The result is in contrast to the case $p = 2$, in which the line contains an infinity of such points (x, y), with square $x^2 + y^2$, whenever it contains one. The latter result follows from the theory of Pell's equation $u^2 - 2z^2 = -v^2$.

Introduction. A theorem of Roth [3] asserts that if θ is a real irrational algebraic number such that $|h/k - \theta| < 1/k^\mu$ for an infinity of fractions h/k, then $\mu \leq 2$. It is an obvious consequence that for any fixed K and $\epsilon > 0$ there can be at most a finite number of fractions h/k for which $|h/k - \theta| < K/k^{2+\epsilon}$.

It is shown here that an infinity of relatively prime, positive integer pairs (x, y) on a line $y = x + v$ such that $x^p + y^p$ is the p-th power of an integer, prime $p \geq 3$, would imply an infinite number of fractions h/k such that $0 < h/k - 2^{1/p} < K/k^{2+\epsilon}$ where K depends only on v and p and $\epsilon = 2/(p - 1)$, which is impossible.

The result is in contrast to the case $p = 2$, where the line $y = x + v$ contains an infinity of such (x, y), with $x^2 + y^2$ a square, whenever it contains one.

1. Fermat's conjecture and Roth's theorem. We prove here the next theorem.

THEOREM. *Let the odd prime $p \geq 3$ and the integer $v \geq 1$ be fixed. Then there are at most a finite number of relatively prime, positive integer pairs (x, y) on the line $y = x + v$ such that $x^p + y^p$ is the p-th power of an integer.*

Proof. Define $b = v/2 \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}$, and let (x, y) be such a pair, with $y = x + 2b$. Setting $a = x + b$, we can write $x = a - b$ and $y = a + b$ in terms of which we would have

$$\begin{aligned} z^p &= (a - b)^p + (a + b)^p \\ &= 2a(a^{p-1} + C_2{}^p a^{p-3} b^2 + \cdots + C_{p-3}{}^p a^2 b^{p-3} + C_{p-1}{}^p b^{p-1}) \equiv 2aQ \end{aligned} \tag{1}$$

from which follow the inequalities

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