COMPLICATIONS OF SEMICONTINUITY IN C^*-ALGEBRA THEORY

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1. Introduction. This paper originated in an effort to correct an error in [13]. Lemma 2.1 in that paper is wrong if the C^*-algebra in question does not have a unit. During the process of restoring the results from [13] it became clear that instead of working with a single class of lower semicontinuous elements (the one denoted by A^m in [13]), one had to deal with several classes, each of which generalizes some properties of the lower semicontinuous functions attached to a commutative C^*-algebra. The analysis of these classes is carried out in Sections 2 and 3 and should have independent interest. In particular we show that although all four classes under consideration might be different they have the same projections (the open projections from [1] and [14]). In Section 4 we study the quasimultipliers of A (those x in A'' for which $axb \in A$ for all a and b in A) and show that these are precisely the elements of A'' that are continuous on the state space [9; Lemma 1]. Finally in Section 5 we show how all the results from [13] can be restored by suitable juggling with the various classes of lower semicontinuous elements.

2. Semicontinuous operators. Given a C^*-algebra A we shall regard it as an algebra of operators on its universal Hilbert space so that the weak closure A'' of A is isomorphic with the second dual of A [8; §12]. For any class M of operators in A'' we denote by M^- the norm closure of M. If $M \subset A_{s.a.}''$ (the self-adjoint part of A''), we denote by M^m the set of elements in $A_{s.a.}''$ which can be obtained as weak limits of monotone increasing nets from M. Let $\tilde{A}$ denote the C^*-algebra obtained by adjoining the unit 1 of A'' to A. We shall be particularly interested in the two classes $(A_{s.a.})^m$ and $(\tilde{A}_{s.a.})^m$ together with their norm closures.

Consider the set $Q = \{\varphi \in A' \mid \|\varphi\| \leq 1, \varphi \geq 0\}$. This is a weak* compact convex set, and the evaluation map ξ on A'' given by $\xi(x)\varphi = \varphi(x)$ for x in A'' and φ in Q is an isometry of $A_{s.a.}''$ onto the set of bounded affine functions on Q vanishing at zero. The image of $A_{s.a.}$ under ξ is the set of continuous affine functions on Q vanishing at zero.

THEOREM 2.1. *Let x be an element of $A_{s.a.}''$. The following conditions are equivalent.*

(i) $x \in ((A_{s.a.})^m)^-$.

(ii) $\xi(x)$ *is lower semicontinuous on* Q.

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