DISTRIBUTIONS OF EXPONENTIAL GROWTH AND THEIR FOURIER TRANSFORMS

RICHARD D. CARMICHAEL

1. Introduction. Sebastião E Silva [13], [14] has defined the space of distributions of exponential growth Λ_∞ and has shown that these distributions arise naturally from the study of functions, denoted $\mathcal{U}_w{}^+$, which are analytic and of slow growth in the half planes $\mathrm{Re}\,(z) > k$ ($\mathrm{Im}\,(z) > k$), $k = 0, 1, 2, \cdots$. In fact $f(z) \in \mathcal{U}_w{}^+$ if and only if $f(z)$ is the Fourier-Laplace transform of an element $V \in \Lambda_\infty$ which has support in $[0, \infty)$. A corresponding result holds for the space $\mathcal{U}_w{}^-$ of analytic functions having slow growth in the half planes $\mathrm{Re}\,(z) < (-k)$ ($\mathrm{Im}\,(z) < (-k)$), $k = 0, 1, 2, \cdots$, where now the corresponding element of Λ_∞ has support in $(-\infty, 0]$. Using these results, Sebastião E Silva has defined the tempered ultra-distributions [14]. Hasumi [9] has extended the results of Sebastião E Silva to n dimensions for the case where the analytic functions are defined in the octants $B_\delta = \{z \in \mathbf{C}^n : \delta_j\,(\mathrm{Im}\,(z_j)) > 0,\ \delta = (\delta_1, \cdots, \delta_n),\ \delta_j = \pm 1,\ j = 1, \cdots, n\}$; so the corresponding element of Λ_∞, via the Fourier-Laplace transform, has support in a product of half lines. (For other interesting results concerning the space of distributions of exponential growth Λ_∞ we refer to Yoshinaga [19] and Zieleźny [20], where the problems of convolution and multipliers in Λ_∞ are discussed.)

Lauwerier [10] has considered a space of analytic functions which are closely connected with the $\mathcal{U}_w{}^+$ and $\mathcal{U}_w{}^-$ functions of Sebastião E Silva. The functions $f(z)$ which are analytic in $\mathrm{Im}\,(z) > 0$ and which satisfy $|f(z)| \leq C_m(1 + |z|)^N$, $\mathrm{Im}\,(z) \geq m > 0$ for all $m > 0$, have been denoted the G^+ functions by Lauwerier [10; p. 162], who has shown that such functions have a distributional boundary value in the weak topology of $\mathbf{Z}'$, the space of ultra-distributions of Gel'fand and Shilov [7]; this boundary value is the Fourier transform of an element in $\mathcal{D}'$ which has support in $[0, \infty)$. Further, the function $f(z) \in G^+$ can be represented as the Fourier-Laplace transform of this $\mathcal{D}'$ element. A similar result holds for the corresponding functions G^- which are analytic in the lower half plane. In Carmichael [2], [3] we have extended the results of Lauwerier concerning distributional boundary values in $\mathbf{Z}'$ to n dimensions, first for the case of the octants and then for the case of arbitrary tubular radial domains, and have obtained several related results concerning the space of tempered distributions $\mathcal{S}'$.

Distributional boundary value theorems concerning analytic functions which have slow growth are important in quantum field theory; in particular the boundary values can be interpreted as vacuum expectation values in a field

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