ENUMERATION OF PERMUTATIONS AND SEQUENCES WITH RESTRICTIONS

L. CARLITZ, RICHARD SCOVILLE AND THERESA VAUGHAN

1. Introduction. Let $\pi = (a_1, a_2, \cdots, a_n)$ denote a permutation of $Z_n = \{1, 2, \cdots, n\}$. A *rise* is a pair of consecutive elements a_i, a_{i+1} with $a_i < a_{i+1}$; a fall is such a pair with $a_i > a_{i+1}$. Thus $r + s = n - 1$. (Note that we are *not* counting the conventional rise on the extreme left or the conventional fall on the extreme right.) If we let $A(r, s)$ denote the number of permutations of Z_n with r rises and s falls, we have the symmetric generating function [3]

$$\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s + 1)!} = \frac{e^x - e^y}{xe^y - ye^x}. \tag{1.1}$$

In the more familiar notation for Eulerian numbers [1]

$$A(r, s) = A_{r+s+1, r+1} = A_{r+s+1, s+1} = A(s, r) \tag{1.2}$$

and (1.1) becomes

$$1 + \sum_{n=0}^{\infty} \frac{x^n}{n!} (\lambda - 1)^{-n} \sum_{k=1}^{n} A_{n,k} \lambda^{k-1} = \frac{1 - \lambda}{e^{(\lambda-1)x} - \lambda}. \tag{1.3}$$

In place of permutations we may consider sequences $\sigma = (a_1, a_2, \cdots, a_m)$, where $a_i \in Z_n$ and the element i occurs exactly e_i times. The n-tuple $\mathbf{e} = (e_1, \cdots, e_n)$ is called the *specification* of the sequence σ. As above, a *rise* is a pair of consecutive elements a_i, a_{i+1} with $a_i < a_{i+1}$, a *fall* is a pair with $a_i > a_{i+1}$, and a *level* is a pair with $a_i = a_{i+1}$. Simon Newcomb's problem [6; Chapter 8] asks for the number of sequences of specification $\mathbf{e}$ with r rises; a solution of the problem was given by Dillon and Roselle [5]. The refined problem of the number of sequences of specification $\mathbf{e}$ with r rises and s falls was discussed in [2].

In the present paper we consider first permutations π of Z_{2n} with the property

$$\overline{\pi(j)} = \pi(\bar{j}), \qquad \bar{j} = 2n - j + 1. \tag{1.4}$$

It is easily verified that the permutations satisfying (1.4) form a group of order $2^n n!$.

The number of such permutations with r rises and s falls is denoted by $A^*(r, s)$. Notice that it follows from the definition that $r + s$ must be odd. It is shown that the numbers $A^*(r, s)$ satisfy the recurrence

$$A^*(r, s) = (r + 1)A^*(r, s - 2) + (s + 1)A^*(r - 2, s) + A^*(r - 1, s - 1). \tag{1.5}$$

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